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# THE KNOWLEDGE OF STUDENTS' MATHEMATICS LEARNING: A CASE STUDY ON PRE-SERVICE TEACHERS IN A NORMAL UNIVERSITY IN CHINA[i]

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**Abstract:**
This study is to provide a better understanding of pre-service teachers' knowledge of students' mathematics learning through the rubrics of trigonometric function. Participants are 6 pre-service teachers from a level normal university. The method of TELT is considered in this study to analyze the factors which can have effect on their understanding. The results show that pre-service teachers always take for granted students' learning and cannot predicate the possible difficulties during their learning and cannot grasp the keystone for breaking through difficulties. It because they cannot know the process of mathematics genesis and development and mathematics essence hidden in the formulas. The results also show that pre-service teachers cannot ascertain the students' mistakes in cognition. And, without the proficiency of mathematics history, pre-service teachers cannot predict and understand students' mathematics learning process.

**摘要**:
本个研究的目的是为了更好地了解职前教师对学生数学学习的了解。研究对象为来自某高师院校的 6 名师范生，通过 TELT 方法，分析了影响师范生对学生数学学习理解的因素。结果表明，职前教师总是把学生的学习心理视为理所当然，不能预测学生在学习过程中可能遇到的困难，不能把握突破困难的重点。究其原因，在于对数学发生发展的过程和隐藏在公式中的数学本质一无所知。研究结果还显示，职前教师无法确定学生错误认知方式的来源。同时，如果不能熟练地掌握应有的数学史知识，职前教师无法预测和确定学生的数学学习心理。

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**Keywords:** China mathematics knowledge, teaching mathematics, mathematical content, mathematical learning, student thinking

**关键词：**中国数学知识；数学教学；数学内容；数学学习；数学思维

## 1. Introduction

It is a prerequisite for effective teaching that teacher should know the students' learning psychology. Educational psychologist Ausubel (1968) stated: the most important single factor influencing learning is what the learner already knows. Ascertain this and teach him accordingly. Teachers should know students' experience and knowledge before teaching. Since the framework of pedagogical content knowledge was created; student knowledge has been paid attention to. Student knowledge and teaching strategies are two important components of pedagogical content knowledge. Shulman (1987) stated that teachers should know their subject matter thoroughly and have knowledge of learners and their characters. Teachers should be aware of the process of learning and understand what a student understand and what is difficult for them to grasp. So, teachers had effective adaptation and tailoring to student characteristics. Student comprehensive and students' knowledge of misconceptions were divided by Grossman (1990) in the framework of student knowledge.

Fennema and Frankle (1992) had the similar viewpoint to Grossman. They divided knowledge of students into subcategories as students' previous and potential difficulties. There were similar sub-categories such as students' knowledge of what they understood, misconceptions in topic area and students' prior knowledge (Schoenfeld, 1998). Park and Oliver (2008) held the point that students' misconceptions played a significant role in shaping pedagogical content knowledge. Teachers must have knowledge about what students know about topic and areas of likely difficulty. The teacher knowledge model of pedagogical content knowledge constructed by An, Kulm and Wu (2004) assert the knowledge of students' misconception and thinking as central to pedagogical content knowledge. They divided student knowledge into such sub-categories as building on student ideas in mathematics, addressing students' misconceptions, engaging students in mathematics learning and promoting students' thinking mathematics. Redefining the model of pedagogical content knowledge, Kilic (2011) defined knowledge of students as teachers' knowledge of what mathematical concepts are difficulties for students to grasp, which concepts typically have misconceptions about, possible sources of students' errors, and how to eliminate those difficulties and misconceptions. The findings revealed pre-service teachers' knowledge of students was intertwined with their knowledge of subject matter and knowledge of pedagogy. Hill, Ball & Schilling (2008) identified a construct of mathematical knowledge for teaching, knowledge of content and student, which include the ability to anticipate student errors, to interpret incomplete student thinking, to predict how students will handle specific tasks, and what students will find and challenge. Hill et al.'s research focused on topic-specific knowledge of students. Student knowledge was

investigated as a connecting prior knowledge to new knowledge, students' mistakes to topic and students' difficulties of the topic (Akkaş & Türnüklü, 2015). Although in the framework of teacher knowledge, the sub-categories of student knowledge differed from different studies, some common characters can be drawn. Focusing on what students know about, identifying errors and difficulties with students, building connection between existing knowledge and new knowledge is the main component of student knowledge. Although the common personal characteristics, such as motivation, ability, learning style, interest, difference of needs, are important for teachers to know students well, the trend of research gradually put focus on the interaction between content and learning psychology.

Some research paid attention to particular component of teacher knowledge of students. Graeber & Tirosh (2008) noted an increasing role for teacher knowledge of students with particular attention to students' thinking. Teacher knowledge of students' thinking advances teaching for understanding (Empson & Jacobs, 2008).Error analysis was a basic and important task in mathematics teaching, and can be used to examining mathematics teacher knowledge of students (Peng & Luo, 2009). Identifying errors ,analyzing reasons for errors, designing approaches for correction, and taking action for correction was an important approach to acquiring knowledge of students' thinking (An & Wu, 2012). Knowledge of students' mathematics learning furnishes specific insights that help teachers gauge how well students understand mathematical conceptions, recognize error patterns and understand possible misconceptions behind the errors, and develop strategies to correct the misconceptions (An & Wu, 2014). Knowledge of students plays an important role in teaching and learning. There were some Chinese experiences. The new century action of Qingpu also pointed that the secret of improving teaching efficacy lies in learning about which knowledge students can easily understand and which knowledge students are apt to making mistakes (Yang, 2007). Teaching should take consider of students' development and keep students in the mind, while not turning a deaf ear to students. In Chinese teachers' words, it is essential that we should attach importance to the student mathematics learning psychology and assessment. Although students' main learning task is to master indirect experience, this learning from books, and the acquisition of indirect experience should base on students themselves' direct life experience. Zhang, an academician of the Chinese academy of sciences, has an opinion of "*seeking conception from student's brain*" (Zhang, 2004). In order to enhance the teaching efficacy, it is necessary not only to study students' general psychological characteristics, such as personality, interest and motivation, but also to pay special attention to psychological characteristics with subject contents. These arguments are consistent with the NRC's opinion that class room teachers engage in inquiry to deepen their understanding of students' thinking (The National Research Council [NRC], 2001). It is no purpose and is inefficient that if teachers' teaching does not consider students 'confusions and cognition obstacles on particular topics and does not construct students' understanding model on students' prior experience and knowledge.

The goal of this study is to acquire pre-service teachers' actual level in knowledge of students on mathematics learning, and to analyze the reason behind the level. And giving evidence-based method and route to enhance the quality of pre-service teachers. The research questions in this study are: (1) What levels are the pre-service teachers in the knowledge of students on mathematics learning during the course of analyzing students' learning psychology on particular topics? (2) What factor makes the pre-service teachers in the knowledge of students on mathematics learning in different levels?

## 2. Theoretical Framework

Both the subject matter and students' characteristics were focused by knowledge of students. One type of framework for knowledge of student put emphasis on content knowledge. Shulman (1987) stated that teachers should grasp subject matter knowledge profoundly and know the process of learning and understand what makes the learning of specific topics easy or difficult. Then, teachers need knowledge of the strategies most likely go be fruitful in assessing students' understanding, addressing students' learning difficulties, and reorganizing students' understanding. In the framework of mathematics teaching (MKT), content knowledge and knowledge about students were integrated, named as knowledge of content and student. Knowledge of content and student (KCS) is an important component of Mathematical Knowledge for teaching (MKT). KCS is content knowledge intertwined with knowledge of how student think about, know, or learn particular content (Hill, & Blunk, et al., 2008). KCS is used in teaching specific content, simultaneously considering how to express the essence of content in an easily understood way and knowing something particular about learners. For instance, if we teach a mathematics formula to students, we should not only know the physical or other origin of a formula and the utility of the formula, but also be aware why students puzzle and have difficulty in correctly applying the formula. KCS contributes to teachers' instruction designation and lays emphasis on the key of the instruction. Another type of framework for knowledge of student put emphasis on student thinking. Knowledge of students is generally defined as knowledge about the characteristics of students and designing instruction to meet the needs of these students according to their characteristics (Fennema & Franke, 1992). According to the Shulman's PCK, Knowledge in students' learning, thinking and ideas is a vital component in the frame of PCK. An, Kulm, & Wu (2004) identified the framework of PCK form the angle of students' thinking. This concept framework pointed out that knowledge on connecting students' prior knowledge with new knowledge, knowing students' conceptual understanding rather than procedures or rules, identifying and eliminating students' misconceptions was key of knowledge of student.

The frame MKT absorbs of the merit of PCK, attaching great importance to KCS.KCS emphasizes understanding and teaching specific topic from the view of students, and do not impose teachers' understanding and opinions on students. KCS put forwards demands of high qualified teachers' knowledge. The general theories of

learning and teaching are just an important reference because these theories of learning and teaching are based on specific subject instances. We insist that mathematics education research should have her own subject characteristic. Both knowledge of content and knowledge about students thinking should be considered in the framework of knowledge of content and student. In fact, teachers' knowledge of subject matter and student thinking had strong influence on their teaching (Morist, Hiebert & Spitzer, 2009). Knowledge of content and student is defined as teaches having knowledge in predicting difficulties and confusions that may be encountered during the process of learning a particular subject, and teachers owning the knowledge in identifying and correcting the root of error and wrong way of students' cognition. KCS reflects how students' understanding of particular content, and possesses distinct discipline characteristics, different from some kinds of general students' psychological researches which have nothing to do with the subject matter. Students' difficulties, errors, misconceptions and so on were result of lack of understanding content.

## 3. Research Method

### 3.1 Subjects and Procedures

The participants were six pre-service teachers from high-level normal university. The researchers acted as trainee instructors and guided the pre-service teachers' teaching practice. The researchers audited their teaching lectures and collaborated textbooks and teaching methods with them. Take the typical sampling method to select 6 pre-service teachers who teach trigonometric function. For convenience, named them as A, B, C, D, E and F respectively.

In this study, the participants have finished a series of mathematics education courses, such as pedagogy of mathematics, micro-teaching train, education psychology and modern education technology. They have some theory knowledge about teaching. In the period of teaching practice, we made a detail plan. All the pre-service teachers should listen to their tutors' lessons and their counterparts' lesson as more as possible and take fielding notes at the same time. Everyone should design his/her lesson plans on one's own. Only after the tutors' scrutiny, the pre-service teachers could give new lessons to students. When finishing the new lessons, it was a routine work to write reflective journal. Every week, we held a meeting to discuss the teaching confusion and doubt and found a suitable way to solve these problems.

### 3.2 Data Collection and Instruments

The research focused on the actual level of pre-service teachers' KCS .The data were collected mainly by interview. Other data were collected from their reflective journal, field notes following in-depth interviews, and research's field notes and classroom observations. Every pre-service teacher firstly answered the interview questions by pen and paper. And then, an in-depth interview was taken in the research's office and recorded under their permission. We collected their reflective journal and field notes

when listening others' classroom teaching, mining data about the interview questions. The research's field notes and classroom observations were also considered. Referring all of the data, we got a pre-service teacher's answer for a survey item. The reliability and validity of data was ensured by using triangulation of data, member checks, and peer examinations.

As part of the Teacher Education and Learning to Teach (TELT) study conducted at Michigan State University, researchers developed a series of survey items to assess teachers' knowledge of subject matter and teachers' knowledge of effective teaching practices in a given content area (Kennedy et al., 1993, Gencturk & Lubienski, 2013). The advantage of TELT can combine a specific conception and the classroom context in which the conception plays an important role, while not directly measure the subject matter knowledge. Referring to the TELT, we develop content-specific instruments to assess pre-service teachers in knowledge of students' learning radian system, trigonometric function of arbitrary angle and the usage of trigonometric formulas. The study literatures show students having difficulty in learning these topics (Blackett & Tall, 1991; Breidenbach & Dubinsky, et al., 1992; Hun, 2004; Rajan & Patil, et al., 1990; Dlice, 2003). That how the pre-service teachers predict and identify students' learning condition reflects the pre-service teachers' KCS. The following question can be cited as an example. The question is that do you know where is easily understood or not easily understood by students in your teaching? How do you conquer the difficult point in the classroom teaching? Please cite an example to illustrate your teaching method. According to the literatures, we collect data through interview, observation, lesson plan and reflective journal around the above mentioned learning topics.

### 3.2 Data Analysis

The researchers followed an inductive data analysis process (Bogden & Biklen, 1992). During this round of analysis, all data were mined in accord with the process of "analyzing-comparing-reanalyzing" to establish inter-coder agreement with regard to instances of KCS. Differences were discussed and looked up the original data until consensus was achieved. Analysis focused on identifying the way that how the pre-service teachers deal with the teaching content according to students' actual state of learning. To attain effective teaching, a teacher should have the ability to predict the possible difficulties or puzzles lying before students and to identify the root and way of the wrong cognition from students. After the classification and coding of the qualitative data, quantification of data was carried out. In order to analyze the actual levels of the pre-service teachers' KCS, referring to a modification of Perkins and Simmons' levels of understanding framework (Kinach, 2002) and mathematical cognitive levels analysis framework, A new frame was proposed to analyze and identify the distribution levels of the pre-service teachers' KCS (Xu & Gu, 2014).

Content-level understanding which refers to knowing about the facts, definitions and algorithms associated the "content" of a subject matter. Associated with the frame are characteristic performances, including recall of facts and correct description of

instances in question in the vocabulary of the domain. The score in this level is assigned 1.

Concept-level understanding which refers to knowing about and experience with the generalized ideas that define, bound, and guide inquiry in a discipline. These are the ideas without which there would be no discipline. For example, without the concept of function, there is no algebra. Associated with the frame are characteristic performances, including identifying patterns and relationships, categorizing the phenomena and processing them into a class. According to the need of a model, the existing concepts, procedures, and properties can be generalized and specialized and the different representations of mathematics concepts and propositions can be transformed from one form into another form. The score in this level is assigned 2.

Problem-solving level understanding which refers to general and domain-specific strategies and heuristic schemas for monitoring one's thinking. Characteristic understanding performances at this level within mathematics include thinking abilities such as finding a pattern, working backwards, solving a similar problem, applying knowledge in different situations, or creating mathematical representations to model physical or social phenomena. The score in this level is assigned 3.

Epistemic-inquiry level understanding which refers to the warrants for evidence in a discipline or the generation of new knowledge that advances thinking in the field. Characteristic understanding performances at this level include knowing about knowledge itself: the sources of knowledge; how it is tested and changes over time; what counts as evidence; and the nature of good explanations and knowing about competing schools of thought within a discipline, the logical structure of the discipline itself, its relation to other fields; knowing about problem posing and theory building and knowing about what is worthy of study and how it should be studied. The score in this level is assigned 4.

And then find the reasons behind the back of data. Analyzing the factors which affect the distribution levels of the pre-service teachers' KCS from the angle of the genesis and development of mathematics, which is a new viewpoint and has attracted the attention of the researchers, such as Clark (2012) and Janvist (2012), etc.

## 4. Results

### A. The qualitative data predicting the difficulties students may encounter in the conception of radian system

We designed a question: Do you know students' difficulties in learning radian system? We collected every pre-service teacher's answers to this question from many ways and integrated all the answers together respectively.

Xiao A thought: *It is not an easy thing to understand radian system conception. I made students understand the conception mainly though the history and practical use. Then by some examples, let them feel the superiority and necessity of the conception.* His tutor gave him some advice on this topic that the start point of its instruction should be the angle system and

told him some reasons. Xiao A agreed with his tutor teacher. Why the start point of teaching radian system is the angle system? He did not have his own viewpoint. His understanding was at concept-level, and the score was 2. Obviously this understanding cannot predict students' learning difficulties.

There came a surprise. Xiao B said that *I felt very puzzled. Why did we introduce the radian system? I did not know the reason. Some teacher introduced this topic by the usefulness of radian system, while my tutor introduced this topic by the rationality. I thought the reason for introducing the radian system was not useful but rational. The same thing can be represented in different ways. There was an example. We could use meter as a unit to measure a man's height, at the same time we could use feet as a unit to measure a man's height too. It was reasonable.* Xiao B emphasized the motivation of introducing the radian system lying in rationality not usefulness. Usefulness is displayed after introducing the radian system. Both of the two systems of measuring angle are rational. The key point lies in which system is more rational. Xiao B predicts students' learning difficulties from the viewpoint of methodology. Following this thinking way, he could predict students' learning difficulties. His understanding was at epistemic-inquiry level, and the score was 4.

There came three similar views. Xiao C said that *students could not understand the radian system well and did not know the use of the radian system. Why should the* $60^0$ *write in the form of* $\frac{\pi}{3}$ *? In students' view, it is just another representation form. Students had no idea of the radian system. I taught this topic by comparing the two systems and gave the definition of the radian from the arc length formula.* Xiao C could not give his students a rational explanation. His understanding was at concept-level and the score was 2.Xiao D audited his tutor's lesson. And he gave his advice for this lesson. *After pointing out the shortage of the angle of system, we could introduce the radian system. We could give some chances for students to know the transformation of the two systems.* Xiao D's viewpoint was that the angle system had some shortages, and the radian system were more useful. His understanding was at concept-level, and the score was 2. Xiao F found that it was difficult for students from students' exercise. Students used to represent an angle with the angle system not with the radian system. I would mention the usefulness of area formula and arc length formula represented by the radian system. The radian system was a rule. Students could master this content by practicing. Xiao F predicted students' learning difficulties from the usefulness of the radian system. But the usefulness cannot answer why it is difficult for students to learn the radian system. His understanding was at concept-level and the score was 2. Before introducing the radian system, usefulness could not count as evidence to explain a concept coming into being. Xiao C, Xiao D and Xiao F could realize this.

Xiao E did not refer to usefulness. Xiao E said that *the establishment of the radian system and its usefulness were difficult for student. I would briefly tell students the background of this topic and grasp the one-to-one correspondence between angle and real number.* Xiao E knew how to solve a geometry problem form the angle of algebra. His understanding was at problem-solving level, and the score was 3. The data can be briefly summarized as Table 1.

**Table 1:** Predicting Students' Difficulties in Arc System

| The Main Points of Answer | Pre-service teacher | Level | Yes or not to predict students' difficulties |
|---|---|---|---|
| According to radian length formula, so, the ration of $l$ and r was only related to the central angle and had nothing to do with the value of r. So, the value of the ration of $l$ and r was looked as the radian of the corresponding central angle. When equals to r, the ration was 1. Thus, the central angle, corresponding to an arc which equals to the radius was looked as 1 radian angle. And this made some formula more concise (usefulness). | Xiao A<br>Xiao C<br>Xiao D<br>Xiao F | 2 | not |
| The meaning of introducing radian system did not lie in usefulness but for which measurement system was more reasonable. | Xiao B | 4 | Yes |
| From the viewpoint of corresponding, the unit 1 of the domain and range of the abstract function should be consistent. | Xiao E | 3 | not |

**B. The qualitative data predicting the difficulties students may encounter in the definition of the trigonometric function**

There are two different ways to define the trigonometric function. One insists that the trigonometric function is a fixed ration reflecting the relation between the angel and edge of a triangle and is an important tool. Others insist that the trigonometric function is a useful tool to study the periodic phenomena, and uses a unit circle to define the trigonometric function. No matter you adopt which definition; teachers should make students understand why the conception was defined in this way or in that way. They should predict students' difficulties in understanding the way of definition.

Xiao A taught this lesson. His experience was that students did not remember the expression of the definition and did not know the way of definition. He described his change in teaching. *After discussion, I gradually knew the way to define the trigonometric function of an acute angle. And we should look a thing from different ways and bridged the learned knowledge with the new knowledge.* After frustration and reflation, Xiao A found students' learning difficulties. Students did not know why we used the rectangular coordinate system to define the trigonometric function of an arbitrary angle. It is a cognitive difficulty because it reflects the change of the research method. His understanding was at epistemic-inquiry level, and the score was 4.

Xiao D gave an explanation from the angle of research method. Xiao D said *if we translated the right triangle to the coordinate system, the relation between the two definitions was clear.* She drew a figure to illustrate her opinion. Xiao D grasped the relation between two definitions. Her understanding was at epistemic-inquiry level, and the score was 4.

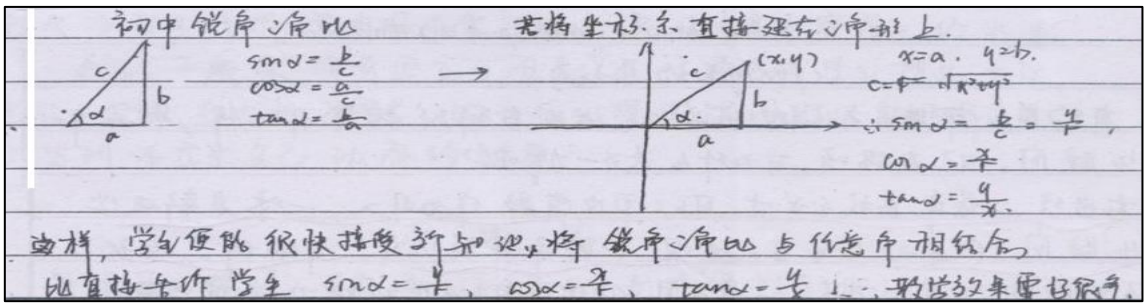

**Figure 1:** Xiao D's illustration

Xiao C seemed to see the difference between the two definitions. The trigonometric function of an acute angle was the special case of the trigonometric function of an arbitrary angle, but the trigonometric function of the arbitrary angle could not be looked as the generalization of the function of an arbitrary angle. Although Xiao C noticed the difference between the two definitions, he could not give the path to help students overcome the difficulties. His understanding was at concept-level, and the score was 2.

Xiao B, Xiao E, and Xiao F seemed to hold the same opinion that the trigonometric function of an acute angle was the special case of the trigonometric function of an arbitrary angle, and the trigonometric function of an arbitrary angle could be looked as the generalization of the trigonometric function of an acute angle. Obviously, this opinion is not to grasp the relation between the two definitions. Their understanding was at content-level, and the score was 1.

The data can be briefly summarized as Table 2.

**Table 2:** Predicting Students' Difficulties in Definition of the Trigonometric Ration

| The Main Points of Answer | Pre-service teacher | Level | Yes or not to predict students' difficulties |
|---|---|---|---|
| From two different viewpoints to see the same thing. For the influent and natural transition, a bridge should connect new knowledge to the learned knowledge. | Xiao A Xiao D | 4 | Yes |
| Feeling the difference of the way to define trigonometric function of an acute angle and an arbitrary angle, the trigonometric function of an acute triangle is a special case of the trigonometric function of an arbitrary angle, was not the development of the trigonometric function of an arbitrary angle. | Xiao C | 2 | Not |
| The o trigonometric function of an arbitrary angle was the development of the trigonometric function an acute angle. | Xiao B Xiao E Xiao F | 1 | Not |

**C. The qualitative data ascertaining the cognitive reasons that students misuse the trigonometric formulas**

The teaching of trigonometric function can easily go astray to teach the operation of symbols and numbers, and the vivid ideas were covered by the operation. It is very important that a teacher can ascertain students' cognitive performance. For this reason, we designed a question.

At the beginning of learning trigonometric function, some students like to use the definition of trigonometric function to do some operations rather than to use the trigonometric formulas of the same angle. How do you think about this phenomenon?

Xiao A held a point that it was a natural phenomenon for the beginners. The beginners could induce these formulas from the definition, because they were familiar with the definition rather than formulas. This explanation didn't get to the heart of the matter. Xiao B had the same opinion with Xiao A. In Xiao B' eyes, it was a natural reaction to new knowledge. They used to apply their acquaint knowledge to solve new problems.

Xiao C gave a reasonable explanation from the angle of formula application. Xiao C thought it was the indispensable period for students and it depended on their cognitive structure. It was common that many methods could be used to solve problems. We praised the students who used the definition, at the same time we should lead them to use new knowledge and helped them compare the advantages and disadvantages of two methods. Students would realize the advantages of the latter.

The rest of the pre-service teachers didn't get to the point. Xiao D's answer was that students might have deep impression on the definition and did not know these formulas well. We should guide them to use new knowledge to solve the old problems. Xiao E thought it was a habit of students and most of students did not attach enough importance to these formulas. If they could find these relationships and used these relationships as more as possible, they would like to use these formulas rather than the definition. Xiao F had two opinions. Students did not have proficiency in these formulas and did not use them flexibly. Students could not bridge the known and the unknown and so they could not choose the right formulas.

The opinions of the pre-service teach can be classified into two kinds. Xiao B could be cited as an example. They thought this was a normal learning phenomenon. Xiao C was another example. They thought this phenomenon could be explained by problem-solving with many methods. Their understanding was at the problem-solving level. From the historical evolution, we can get enlightenment. The independent variable----angle should be emphasized, which is the important characteristic of trigonometric function. These formulas representing relation of the same angle is to emphasize this important characteristic. It is the key for teaching. The data can be briefly summarized as Table 3.

**Table 3:** Diagnosing the Reason of Students' Misusing Formula

| The Main Points of Answer | Pre-service teacher | Level |
|---|---|---|
| During the turning period from middle school to high school, it was a natural choice when students use the unfamiliar new knowledge to solve problems. | Xiao A | 1 |

| | | |
|---|---|---|
| It was a natural phenomenon that students are familiar with old things and are unfamiliar with new thing. | Xiao B | 1 |
| It was a phenomenon of solving a problem with many ways. If you forgot the relationship between the trigonometric function of the same angle, you could try to use this method. | Xiao C | 3 |
| Xiao D held the same viewpoint with Xiao B. | Xiao D | 1 |
| It was the result that students pay less attention to the relations between of the trigonometric function of the same angle. More times to use this relation, more less times to appear this phenomenon. | Xiao E | 1 |
| There was a conflict between previous and later viewpoint. When students rushed to show his discovery to Xiao F, he held the viewpoint that the student went back to use definition to solve problem, which should be encouraged and should select an appropriate method. Later, the pre-service teacher changed his assessment. He gave an explanation that the linkage between the new and old knowledge did not connect in students' mind due to not knowing well the new knowledge from the perspective of learning psychology. | Xiao F | 1 |

What can be seen from the above results, the pre-service teachers did not have the ability to "predict" students' cognitive difficulty nor to "ascertain" the reason of students' wrong cognitive models. This is consistent with the characteristics of the pre-service teachers' learning. Always they pay much more attention to understand the subject content deeply, without considering the integration of subject content and learning psychology. The result supported the earlier studies on teachers' knowledge of students (e.g. Ball, Thames & Phelps, 2008; Kilic, 2011).

## 5. Discussion

Analyzing the evolution of the way of defining concepts and heat thinking and cognitive conflicts behind Cognitive outcomes contributes to predicting students' cognitive difficulties

Grasping the cognitive difficulty of conception helps teaching. Take the teaching of the radian system as an example. There are many factors influence students' cognition, such as the dimension of measurement, different way of measuring a same quantity, how to understand Pi, the formula of arc length in junior middle school textbook, trigonometric function and the physical model of trigonometric function, and so on, which makes the radian system is difficult to learn in the teaching of triangle function.

The key to breakthrough of difficult point, making it easy for students, lies in finding the cognitive node. Contrasting the teaching materials now and before, an important statement was deleted in the textbook of the second curriculum reform. The statement is that "in plane geometry, the circumference of a circle was divided into 360 equal parts, each called 1 degree of arc, the central angle formed by 1 degree of arc is called 1 degree angle." Looking up mathematical history, we can infer it is full of contingency and subjectivity that the circumference of a circle is divided into 360 equal pieces. Why not divide it into 370 equal parts? In fact, the ancient Chinese divide the circumference of a circle into pieces. In terms of the radian system, a series of

inconvenience of angle system is due to the arbitrary decision of a circumference. If the length of radius is taken as a unit to measure the circumference of a circle, then the circumference of a circle can only be divided into units no matter in what condition, which is a natural law and not to bow to man's will and can be used to explain why the radian system is superior to the angle system in theory. From the above analysis, the common basis of the radian system and the angle system is how to equally divide a circumference of a circle. Dividing the circumference of a circle into 360 equal pieces is one of historical rules, while the radian system reflects an objective law, more scientific and more reasonable. The basic character of angle system is to use a special angle to measure an angle, this is to say, measuring an angle by oneself; while the radian system uses a segment to measure an angle, in virtue of other quantity to measure an angle, this is to say, measuring an angle by others. Using different units to measure the same circumference of a circle naturally forms a relationship arc=360 degree, just liking using km and m to measure a distance of 1000 meters, there always is 1 km =1000 m. So, the transition from the angle system to the radian system is how to divide the circumference of circle more reasonably and more scientifically, which is a key point for teaching. This key point can be as the basis of teaching design according to the subject content. Grasping the key point, arrange proper teaching sequence to illustrate the underlying principle clearly and help students break through difficulty in cognition. The reason why the radian system is introduced into textbooks does not lie in many advantages the radian system owns. A teacher who holds this point obviously puts horse before the carriage because when the scientific radian system became a mathematical conception, the radian system owns many merits from then on. Inspired by these analyses, Xiao B said she would look up a lot of books, and just only reading the textbook, she was unable to teach. It is meaningless that a teacher teaches without her/his understanding. Most of the pre-service teacher put emphasis on usefulness of the radian system, so they could not predict difficulties students would meet. The level of KCS was low.

Analyzing the subject deeply to see the evolution way of defining conception and the chain effects, and discovering the heat thinking and cognitive conflict behind cognitive outcomes contribute to predict students' cognitive difficulties because present students may encounter difficulties ever occurring in history (Fauvel & Van Maanen, 2000). The pre-service teachers always cannot grasp the key point in predicting the possible difficulties students may encounter, sometimes just as "*cannot see the woods for the trees*".

The formation process of subject content has experienced the evolution and competition of cognitive perspective, full of confusion and difficult choices, knowing nothing about which a teacher cannot know students' psychology well and cannot acquire the pedagogy of teaching

It is a prerequisite to determine scientifically the direction of teaching that a teacher should accurately grasp students' learning psychology. From pre-service teachers' perspective, the trigonometric function of an acute angle is same to the trigonometric functions of an arbitrary angle. The understanding of the conception of

trigonometric function has experience a process of "from static to dynamic". The textbooks of 1960s held the viewpoint that the trigonometric function was a fixed ration reflecting the relationship between edges and angles of a triangle and were an important tool to probe the relationship between edges and angles of a triangle; while now the trigonometric function is regarded as a basic model to describe the periodic phenomena. Someone puts forward an advice that a teacher should adopt the process of "from dynamic to static" in the teaching of the conception of the trigonometric function ,which uses a unit circle in the rectangular coordinate system to highlight the dynamic characteristic as a function. They thought that this processing method helps students to further understand the significance of the trigonometric function value by the way of ration and thus facilitate students to have a better understanding of the trigonometric function. However, many teachers do not agree with the expert's opinion. In classroom teaching, they adopt the process of "from static to dynamic" that is to review the trigonometric function of an acute angle and then generalizes it to an arbitrary angle. Here, no matter in what order, a teacher should accurately predict the learning difficulties students may encounter and make a clear teaching orientation, so as to make a decision that which arrangement of teaching sequence can fit students' actual cognition level. A pre-service teacher did not notice it and had been embarrassed. His teaching process was as following: firstly review the trigonometric function of an acute angle, and then put the terminal side of an angle in the second quadrant, select a point in the terminal side of the angle and drew the vertical line of the horizontal axis from the point, thus forming a right triangle; thirdly, define the trigonometric function of an arbitrary angle as the way of defining the trigonometric function of an acute angle; finally, he asked students "Do you understand?" Students answered together "Do not understand." why did this happen? Because when students began to learn the trigonometric function of an arbitrary angle, they may encounter difficulties of the way of defining and the change of cognitive perspective, rather than others.

The transition in epistemology in history is where students' cognition difficulties lie in, which projects onto pedagogy and then demands to bridge the previous and subsequent content. It was a milestone in the development of the trigonometric function that the coordinate method was used to study the trigonometric function. An example is always cited that sine is no longer a directed line segment and became a numerical value or a ration, i.e. sine is the vertical coordinate of a point on the unit circle. Based on these changes of epistemology, the trigonometric function of an arbitrary angle is not regarded as a generalization of the trigonometric function of an acute angle formally. There is a big difference between the two ways of definition. Although the pedagogy "from dynamic to static" sees the changes brought to the trigonometric function by the new research method, it is a backward-to-forward approach by logic after the outcome came into being. The pedagogy" from static to dynamic" obviously sees the consistency in formality. The two methods have some rationality, but are not completely reasonable. The way of definition by ration is a geometry method, while the way of definition by coordination is an analytic method. In history, the transition period from the geometry method to the

analytic method did not happen overnight. These turning points in history are the difficulties for students, to which teachers should pay more attention. Pre-service teachers, such Xiao A and Xiao D, acquire the teaching knowledge by reflection in practice. In fact, if they have knowledge of history on some topic, they could accurately predict students' difficulties as well.

It is a good way to grasp the essence of teaching on the basis of analyzing the subject content. Although someone says that "there is no routine method for teaching, the key lies in knowing the teaching method", where is the best way? Someone says the best way is to adopt flexible teaching means and methods according to different teaching goals and the differences among students of different levels. That sounds good, but is too abstract. The teaching of one subject is a practical work, not just a sitting pontificating. The first important work is to have a clear and accurate knowing of learners' psychology on specific content. Subject matter content is the outcome of human cognitive style and process in solving problems, during which it experienced the evolution and competition of cognitive perspective, full of confusion and difficult choices. If a teacher does not know the history, he/she will not know present students and does not have individual thought and concern about pedagogy.

Whether a teacher can correctly ascertain the cause of students' wrong cognitive style, it depends on how a teacher understands content knowledge. It is prerequisite that a teacher should accurately ascertain the root of students' incorrect cognition and then guide to the point. During the period of making an investigation, a student wrote a reflection after a test, very shocking.

*This is the worst test since I enter high school. When I calm down and set out to rectify the wrong test questions. I feel it is not as difficult as I used to think. It is not difficult to gain a good score as long as I am proficient in the formula and careful in the exam. I have to admit that I do not grasp the formulas to the full extent. When I began to solve a question, I do not know where the starting point is and just mechanically use the formulas and eventually led to the result of "more haste, less speed". It seems that I should study hard at ordinary times. Everything will be all right. Come on!*

This student's learning attitude was very active. But there were some actions against cognitive rules. A pre-service teacher did not point out and just wrote "continue to work hard, and progress will fall on you" on the examination paper. The pre-service teacher did not make correct judgment on the way of students' thinking. Students' cramming was inevitable to master the learning content. It is important for pre-service teachers to make correct judgment on students' cognitive performance.

It is feasible to confirm the nature of students' cognitive understanding from the angle of history. Actually, just from the viewpoint of identical transformation, the mathematical levels of many trigonometric identities do not surpass the mathematical level in the 17th century in all. The trigonometric transform is not difficult, as long as you can grasp the transformation of "angle, function name and structure". It is an effective way to implement triangular transformation with sine or cosine to represent secant or cosecant and with the angles in the conclusion to represent the angles in the condition, as

the high school teachers summarize. In the historical evolution process of the trigonometric functions, the geometric method was used firstly and then was Viete's algebraic method. Since calculus was invented, the analytical method can also be used in the trigonometric function. Euler made a clear distinction between the trigonometric function and the directed line segment of the trigonometric function, which made all the triangle formulas, were deduced logically from the definition of the trigonometric function, separating the trigonometric function from geometry. Enlightenment can be drawn from the historical evolution: emphasizing the angle as the independent variable of the trigonometric function is to highlight the characteristic of the trigonometric function as a function; and emphasizing the name and structure of the function is just the algebra research method used on triangular transformation. It is necessary for students to grasp the transformation of angle, function name and structure during the teaching of classroom exercise. But it is possible to fall into a self-reproduced system, named by the great Physicist Feynman, in which we just know how to do transformation and do not know the ideas behind the transformation. Under such education system, people who have passed all kinds of entrance of exams teach other people how to pass exams; no one tries to understand the essence of knowledge. This kind of phenomenon is widespread. That the above mentioned pre-service teachers understand the trigonometric function almost from the perspective of problem-solving can be taken as an example. They taught the profound change of thoughts into a routine technique. It is important for the pre-service teachers to ascertain the essence of students' understanding from the perspective of historical development.

The pre-service teachers felt it necessary to comprehend the understandings of students in the practice of teaching. Xiao A's experience was that it was effective to prepare lesson from the standpoint of student. Otherwise, inappropriate questions and diction would appear and brought difficulty to students' understanding. The transition from one step to another should be natural and fluent and could attract students' attention to what they were going to do. Secondly, design effective activity for students and let students know what they were doing. The content of activity must be suitable for students' cognition levels. The no-key content did not consist of the carrier of the activities. The amounts of activities should be appropriate and ensure students to complete. Xiao B's experience was that why there were so big gap between students' reaction and my anticipation. Maybe there were some wrongs in my question to lead students or I did not consider the problem from the standpoint of students. I could clearly feel I was short of ability in teaching from students' response. Xiao C drew a lesson from a fail classroom teaching and realized the teaching prince "teaching in accordance with students' aptitude". The most importance of classroom teaching was to adjust flexibly according to students' reaction. Xiao D's experience was that teaching was just like a show and if the teacher was the only role in the stage, it was likely to be unsuccessful. If the major actor could change from the teacher to students, it should be more successful! Xiao E's experience was that if I wanted to teach a lesson well, I would consider questions and reflected my understanding of knowledge from the angle of students and gave

teaching according to the way of students' thinking. Only Xiao F was an exception, who did not obviously realize the necessity of understanding students' thinking.

To learn well about students' cognition psychology bases on the deep and critical understanding of subject content. Teacher's ascertaining of the source of students' wrong cognitive model has to experience three stages: collecting data, evaluating data (estimating the authenticity and accuracy, judging normality or abnormality) and analyzing, reasoning, judging (based on assessment, and then analysis, associate, synthesize, reason, at last make a decision). Whether a teacher can correctly ascertain students' learning situation it depends on how a teacher understands the subject content, the teaching and students' learning psychology. It is impossible that a teacher who cannot understand the teaching content well can teach students to master the content well. Depaepe, Vershaffel & Kelchtermans (2013) pointed the concept MKT resulted from an attempt to refine and empirically validate PCK .Whereas PK and CK are distinct categories in Shulman's conceptualization of teachers' knowledge base, PCK and CK are integrated within one overarching category of knowledge. MKT holds the point that CK has an important effect on PCK. If teachers understand CK thoroughly, his/her view on education can improve to a higher level.

### 5.1 Limitation Future Study

As emphasized throughout this article, the empirical results in this study are intended to be illustrative, not definitive. Replicating this approach with other more samples of teachers is to acquire more general findings. We did not describe how the pre-service teachers' individual experience, context and beliefs were related to the pre-service teachers' SCK. If the psychometric properties of instruments have been more firmly established, more fruitfully results can be acquired in the domain of SCK. Additionally, although we have been careful to avoid casual analysis of the pre-service teachers' kinds of data, there is a potential to make a subjective conclusion in the frame of measurement methodology and research design.

We note the complexity of SCK and always pay attention to good interview item for SCK.A measurement tool that more effectively measure SCK, while also remaining feasible to implement in practice, would be a substantial contribution to the research on SCK. A fine content analysis method should be developed. These are the key challenges to be addressed by future measurement research.

## 6. Conclusion

The pre-service teachers could not accurately predict the learning difficulty students may encounter because they lacked some knowledge on the development of mathematics. For example, they always put emphasis on the usefulness and convenience brought by the concept rather than on the necessity to create concept, thus they could not catch the key to break through the difficulty. The pre-service teachers always ascertain the students' cognitive performance on an assumption or a conjecture from the perspective of problem-

solving and hold the point that why students could not acquire understanding was due to lack of sufficient operational training.

When the pre-service teachers were "predicting" and "ascertaining" the error source of students' cognitive performance, they always taken for granted students' learning psychology or attribute the reason to low efficient teaching. As a matter fact they had no knowledge on the evolution process of certain knowledge and the difficulties and setbacks happened in the evolution process. The pre-service teachers did not realize that the formation of a conception and a method often have experienced a long evolution process and students had difficulty in recognize the essence of knowledge, which was caused by the formal characteristics of mathematics. The research motivation of the problem and the nature of the problem buried in the formal symbol, always puzzled students. And some difficulties in cognition are always reflected in the implicit or explicit form, which exhibits the cognitive historical similarity between mathematicians and students. It is a big challenge for the pre-service teachers to accurately "predict" and "ascertain" students' learning psychology. The history of mathematic may provide a way out of the dilemma.

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