European Journal of Education Studies
ISSN: 2501 - 1111
ISSN-L: 2501 - 1111
Available on-line at: www.oapub.org/edu

doi: 10.5281/zenodo.3692059 Volume 6 | Issue 12 | 2020

DEVELOPING MATHEMATICAL THINKING THROUGH THE ACTIVITY BASED HEURISTIC APPROACH: A CASE OF MAKING CONNECTIONS BETWEEN PATTERNS, SEQUENCES AND GRAPHS

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Abstract:
Teaching approaches is a key factor that contributes to the improvement of learner achievement. Moreover, learning mathematics through activities helps learners to develop mathematical thinking which is the foundation for development and a basis for sustainable development in mathematics. In this study, 77 third year pre-service mathematics teachers at Makerere University were provided with an experience of utilizing the activity based heuristic approach in making connections between patterns, sequences and graphs with the intention of developing their mathematical thinking. The participants were subjected to a whole-class discussion after which they reflected on the discussion. The whole-class discussion was videotaped and data were collected through observation of the video and making reflective notes. Having used observation narratives and thematic analysis of reflections from the whole-class discussion through the lens of the conversational framework, it was observed that learning mathematics through activities helps learners to develop mathematical thinking which is the foundation for their development and a basis for their sustainable development in mathematics. The learners are engaged in the learning process and besides, develop several other competences like communication, creativity and innovation, presentation and working in a team. From the pre-service teachers' perspective, the major implication of the activity based heuristic approach is the need for an exigent change of the pedagogical training approaches for Makerere University's pre-service mathematics teachers.

Keywords: activity based heuristic approach, Makerere University, mathematical thinking, pre-service teachers, and teaching and learning mathematics

1. Introduction

Mathematics is one of the most important subjects (Pokhrel, 2018) in the Uganda school curriculum and consequently, it remains compulsory for all learners at primary level and

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secondary school ordinary level ("O" level) according to the National Curriculum Development Centre (NCDC) (NCDC, 2018). Even for the national grading system, each learner must have passed English and Mathematics to maintain a first-grade score, otherwise, irrespective of how well they have passed the other curricular subjects, the learners score a second grade (Uwezo, 2016). Although the Government has reserved mathematics in a good place in the curriculum, mathematics achievement among learners in schools continues to be low (Nangonzi, 2019; Nabatte, 2019). Teachers still use the traditional replicative teaching approaches in the teaching and learning of mathematics, focusing majorly on passing examinations (Mafabi, 2018; Uganda Technology and Management University, 2016). Conversely, students learn mathematics to pass examinations (Ngware, Hungi, Mahuro, Mutsiya & Abuya, 2016). This indicates that the teaching and learning process of mathematics is guided by the nature of examinations, especially the Primary Leaving, Uganda Certificate of Education and Uganda Advanced Certificate of Education national examinations at the primary, lower secondary and upper secondary levels of education respectively. During examinations, students are asked to solve problems in a paper-pen test format because their mathematics teaching and learning is mostly for supporting the ability to solve problems in a 'do as I have said and done' design, which approach is creating challenges for the 21st century learners because they are not able to logically think or reason beyond laying step-by-step solutions to a given problem (Batiibwe, 2019).

The teacher-centered and didactic instruction only allows for information to flow from the teacher to the learners (Ng'ambi, Gachago, Ivala, Bozalek, & Watters, 2014) and the learners end up doing very many exercises of the same concepts, hence cramming the procedures that they can reproduce when writing examinations so to pass. Although, some learners are able to pass the examinations, this passing could be challenging given that the learners lack high order thinking skills namely application, analysis, evaluation and creation according to Bloom's Digital Taxonomy (Churches, 2008). Thus, passing examinations does not necessarily mean that learning has taken place. Adams, Cummins, Davis, Freeman, Hall Giesinger and Ananthanarayanan (2017) maintain that learning is a social construct which takes account of activities immersed about placing the learner at the center. According to them, learning should be about interaction, working in groups and developing solutions to real world challenges. To this effect, Sharples et al. (2016), advocate for collaborative learning, which besides developing 21st century skills such as collaboration, critical thinking, creativity and communication, also develops intrapersonal skills like persistence and self-regulation. While Siemens (2005) posits that learning is a process of creating connections and articulating a network with nodes and relationships, Vygotsky (1978) affirms that social construction of knowledge depends on communicating gradual ideas through a described dialogue in which students learn how to think. Thus, we need to realize that traditional teaching approaches are no more useful than passing examinations and embrace approaches that prepare learners for the 21st century skills whereby they need to learn to do things by themselves.

According to Polya (1973), the principle of learning by doing is termed the heuristic approach of the teaching and learning process. In heuristic methods of teaching and learning, students are discoverers as compared to the traditional methods where they are listeners of information and are simply being told about things. Pokhrel (2018) affirms that the heuristic method of teaching mathematics enhances students' development of mathematical thinking. Hence, mathematical thinking is an important goal for mathematics given its ability to support sustainable mathematics learning and its emphasis explains students' good outstanding performance in international assessment, for example in China (Li, Mok & Cao, 2019). However, Singh, Teoh, Cheong, Rasid, Kor and Nasir (2018) contend that there is a common misconception that doing mathematics is the same as getting involved in mathematical thinking. According to them, mathematics is a thinking activity that should be taught and learnt by utilizing activity based heuristic approaches. Activity based learning is a process whereby learners are actively engaged in the learning process by performing some hands-on experimentation and activities and involves students in doing things and thinking about the things they are doing (Bonwell & Eison, 1991). Festus (2013) states that the strategies of achieving activity based learning in the mathematics classroom include "*discovery approach of teaching, appropriate practical work, use of teaching aids, cooperative learning or small group learning, and discussion in class*" (p. 8) and all these are linked with activity based mathematics instruction (Pokhrel, 2018). Consequently, activity based learning requires active problem-solving in finding patterns in the information given through one's own investigation and analysis (D'souza, 2016).

Although there are various strategies in the activity based heuristic approach, practical activities were selected for this study. According to Simpson (2003), practical activities are skill oriented approaches in learning mathematics as well as acquiring skills such as practical calculations and estimation; are related to involvement with physical objects and very helpful in gaining firsthand experience; and can be developed in teaching different areas of mathematics. Thus, the overarching goal is to help students develop mathematical competence, in other words, the ability to understand, judge, do and use mathematics across a variety mathematical situations (Nepal, 2016). Further, Pokhrel (2018) observed that classroom practical activities help in "*deepening mathematical understanding and strengthens mathematical skills … and students are motivated to do and learn mathematics*" (p. 47). On the same, learners get actively involved in the process of learning and learning is more learner-centred. Thus, use of classroom practical activities during the teaching and learning of mathematics can develop students' mathematical thinking.

Although teachers' understanding of students' mathematical thinking influence their classroom instruction and hence students' learning, there is far less research on how teachers acquire their knowledge of students' mathematical thinking (Zhu, Yu & Cai, 2018). Further, the Uganda mathematics teacher education has no provision in their curriculum for the secondary school mathematics content that the pre-service mathematics teachers are expected to teach when they qualify as in-service teachers. Even though during their preparation, the pre-service mathematics teachers take three mathematics teaching methods course units namely teaching and learning mathematics, curriculum theory and contemporary issues in mathematics education and assessment, evaluation, resources and materials development in mathematics, these are generally taught using PowerPoint presentation (Batiibwe, Bakkabulindi and Mango, 2018) in a traditional replicative approach and moreover in the absence of the actual secondary school content that they will have to teach after their preparation as secondary school teachers. Basing on Zhu *et al.*'s (2018) and Batiibwe *et al.*'s (2018) submission, the major concern of this paper is to share the experiences of pre-service mathematics teachers in developing mathematical thinking through the activity based heuristic approach, utilizing a mathematical case of making connections between patterns, sequences and graphs. Therefore, the specific objectives of this study were to:

1) Provide a basic experience of activity based learning among pre-service mathematics teachers
2) Develop pre-service mathematics teachers' mathematical thinking through activity based learning in making connections between patterns, sequences and graphs
3) Establish the pre-service mathematics teachers' perceptions on the implication (s) of the activity based heuristic approach to teaching and learning mathematics

2. Literature Review

Several scholars, for example, Delima, Rahmah and Akbar (2018); Delima (2017); Kargar, Tarmizi, and Bayat (2010); Li, Mok and Cao (2019); Nepal (2016); Pokhrel (2018); Singh, Teoh, Cheong, Rasid, Kor and Nasir (2018); Yorulmaz, Altintas and Sidekli (2017); and Zhu, Yu and Cai (2018) have attempted to research on mathematical thinking and its development among learners and its relationship to other mathematics concepts such as achievement and anxiety in different contexts.

Delima *et al.* (2018) analyzed students' mathematical thinking based on their mathematics self-concept (MSC). They further aimed at determining whether there were differences in mathematical thinking between students who had positive MSC and those who had negative MSC and to know how much influence MSC had on the students' mathematical thinking. They used four indicators of students' mathematical thinking namely specializing, generalizing, conjecturing and convincing. While for specializing, Delima *et al.* concentrated on students trying problems by looking at the example and paying attention to a simple case, they focused on students' ability to look for patterns and relationships on the part of generalizing. Regarding conjecturing, they centered on how students predicted relationships and results while for convincing, they concentrated on the students' ability in finding and communicating the reason why something was right. They collected data from 31 senior high school students in Subang, West Java in Indonesia through a test of students' mathematical thinking and MSC questionnaire. Using descriptions and Mann-Whitney U test, they found that students who had positive MSC had good mathematical thinking abilities and significant differences in mathematical thinking between students who had positive MSC and those who had negative MSC respectively. Further, they found the variables of MSC to give positive and big influence on the students' mathematical thinking.

Delima (2017) investigated the relationship between problem-solving ability and students' mathematical thinking and how strong problem solving ability affected students' mathematical thinking. Delima collected data from 13 students of mathematics education and 20 students of information systems at a university in Indonesia using problem solving ability and mathematical thinking tests. Although Delima did not reveal how she analyzed data, she found that there was an influence of problem solving ability on students' mathematical thinking for both groups of students, however, the influence was stronger for the mathematics education students because almost 75% of their university courses involved problem-solving activities as compared to the 10% of those of the information systems students. And as a result, Delima indicated that problem solving ability enhanced mathematical thinking.

Kargar *et al*. (2010) set out to establish the relationship between mathematical thinking, mathematics anxiety and mathematical attitudes among university students. They used a 60-item questionnaire that included mathematical thinking, mathematics anxiety and mathematics attitudes rating scales to collect data from 203 university students from a public university in Malaysia. Having employed correlation analysis to analyze their data, they found a significant high positive correlation between mathematical thinking and mathematics attitudes. Further, they found a negative moderate correlation between mathematical thinking and mathematics anxiety and a negative correlation between mathematics anxiety and mathematics attitude. Thus, Kargar *et al*. indicated that the level of mathematics anxiety was related to mathematical thinking and mathematics attitudes.

Li *et al*. (2019) aimed at providing a comprehensive understanding of mathematical thinking based on the statements of mathematical thinking in Chinese official documents. They reviewed definitions, descriptions and explanations from a historical perspective by way of document analysis. Their analysis indicated that mathematical thinking places more emphasis on the process of mathematical methods application in problem-solving such as the method of combination of symbolic and graphic mathematics and focuses on making the concept more understandable.

Nepal (2016) sought to find out the level of mathematical thinking and the mathematics achievement of the students of grade X in Nepal gender and location wise. A total of 400 students from 40 schools in three districts namely Sindhupalchok, Kathmandu and Mahottari participated in the study. Of the 400, 200 of these were females while the other 200 were males and 150 of them were based in rural schools while the remaining 250 were from urban schools. All of these 400 students were subjected to mathematical thinking and mathematics achievement tests. Having analyzed the test scores using t-test, he found out that there was no significant difference on the level of mathematical thinking and the mathematics achievement between male and female and there was a significant difference between rural and urban students on the level of mathematical thinking and mathematics achievement. They indicated that their findings were applicable to improving teaching strategies and creating new plans to make better performance of the students from rural areas.

Pokhrel (2018) aimed at providing experiences in addressing the 21st century skills through activity based mathematics instruction among students in a school in Kathmandu Valley in Nepal. Pokhrel designed and implemented different games, practical oriented games, math lab activities, exhibition and projects. Pokhrel collected data by spending a total of 33 days of 45 minutes each in different grades from grade one to 10 observing the implementation of the activities. He also supported the classroom teacher in the activities and he interviewed 25 students in the process. Pokhrel further utilized a full day in observing and interacting with the students on the project exhibition day. Having used thematic analysis, reflections and narratives, Pokhrel observed that mathematics learning through activities is helpful for learning mathematics as well as all-around development of students because students are engaged in the learning process. In addition, he found that students also learn different skills such as working in a team, leading group members, communication and presentation, creativity among others, the skills required for the 21st century; develop a positive attitude towards mathematics to some extent; are motivated to do and learn mathematics; deepen their mathematical understanding; and strengthen their mathematical skills.

Singh *et al*. (2018) endeavored to review the effects of a problem-solving heuristic application technique on learners' mathematical thinking development. Their two phase study analyzed data utilizing a descriptive design and experimental design. In the first phase, 660 high school leavers with an A grade in the mathematics national examination registered with a College to undergo a two-year diploma engineering programme were administered a mathematical thinking test. Then 54 of these students participated in a seven-week pre-post experimental design to investigate the impact of heuristics application on their development of mathematical thinking. These students' post-test score was also compared with a batch of 120 third-year university students majoring in STEM related courses where all these students had taken at least five university level related mathematics courses. The results indicated that the high school leavers' grades obtained in the national examination was not translated into their mathematical thinking process. Secondly, the results depicted a significant increase in the mathematical thinking post-test score among the students who underwent a seven-week pre-post problem solving heuristic treatment. Thirdly, the students involved in the heuristic application treatment performed better than the third year students in the mathematical thinking test.

Yorulmaz *et al*. (2017) sought to investigate the effects of mathematical thinking states of form teachers on their mathematics teaching anxieties. They collected data from 194 form teachers working in state schools of Bagcilar District of the Instanbul province in Turkey, using a mathematical thinking scale and an anxiety scale for mathematics teaching anxiety. They analyzed data using multiple linear regression and found that the form teachers had high mathematical thinking scores and low anxiety scores. They also found a low degree, negative and significant correlation between the mathematical thinking and anxiety of form teachers regarding mathematics teaching and that mathematical thinking had an effect on anxiety among the form teachers regarding mathematics teaching.

Zhu *et al*. (2018) compared the differences between expert and non-expert mathematics teachers on their behaviors and perceptions related to understanding students' mathematical thinking. They collected data from 554 Chinese elementary mathematics teachers through a survey. Having analyzed the data using descriptive statistics and chi-square tests, they found that the non-expert teachers claimed they did not rely on prior teaching experiences to understand the students' mathematical thinking during the teaching and learning process because they did not know how to. Further, their comparison revealed that significantly more expert elementary mathematics teachers attempted to understand students' thinking from a variety of perspectives before making the necessary adjustments to their predetermined teaching plans than did non-expert teachers.

From the nine reviewed articles, two major gaps emerged. First, although all the nine studies had interest in mathematical thinking, apart from Li *et al*. (2019) who reviewed literature on understanding mathematical thinking by way of document analysis, and Pokhrel (2018) who provided experiences for developing mathematical thinking through activity based mathematics instruction, the rest of the studies looked at relationships between mathematical thinking and other mathematics concepts like mathematics anxiety, mathematics achievement, mathematics self-concept among others. In addition, all these relationships were tested through data collected from mathematical thinking tests/ scales and/ or mathematical thinking questionnaires/ surveys alongside with other mathematical concepts tests/ scales and questionnaires/ surveys. Even though only Pokhrel (2018) attempted to provide experiences of developing mathematical thinking through activity based mathematics instruction, he did not report the mathematical process narrations involved in developing mathematical thinking in light of a real classroom setting. Secondly, all the nine studies did not employ theoretical frameworks to ground their studies yet theoretical frameworks play an important role in explaining, predicting and mastering phenomena such as relationships, events or the behavior (Abraham, 2008; Adom, Emad, & Adu-Agyem, 2018). Thus, this study filled these major gaps by providing the mathematical process of developing mathematical thinking in a real classroom setting and further, employed the conversational framework as the theoretical framework that guided the research.

3. Theoretical Framework

In this study, Laurillard's Conversational Framework (2002), as displayed in Figure 1, was adopted to show the understanding of the communication and collaboration processes between the teacher and the student during the teaching and learning process. The framework has four significant elements namely teacher's concept; teacher's constructed learning environment; student's concept; and student's action. There is great prominence on the teaching and learning process which is exhibited by underscoring the students' process of understanding the learning content amid their own reflections (Laurillard, 2008) and alteration of information regarding the given mathematical activities, in addition to the feedback from the teacher.

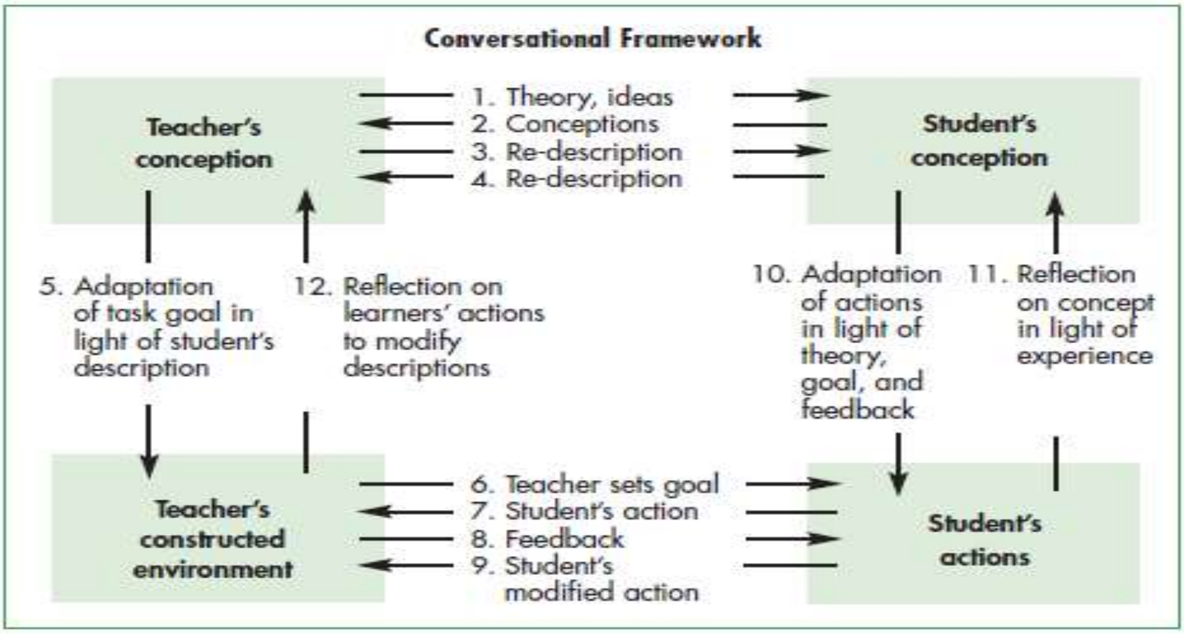

Figure 1: Conversational Framework (Laurillard, 2002)

With the interaction and feedback obtained from the teacher, the learners can better understand the concepts and the objectives of the mathematical activities at hand. They can then work on their tasks and further attain experience in problem-solving, critical thinking and communication skills according to Bloom's taxonomy (Churches, 2008). The framework necessitates the learners to continually go round this cycle of attending, questioning, practicing, adapting their actions, using feedback, reflecting, and articulating their ideas (Laurillard, 2012) and this is what agitates the conversation between the teacher and learner.

4. Methodology

4.1 Research Paradigm and Design

The study intended to provide third year pre-service mathematics teachers at Makerere University with an experience of using actual secondary school mathematics in making connections between mathematical topics namely patterns, sequences and graphs using mathematical activities. The study was based on the interpretivist research design. According to Taylor and Bogdan (1998), an interpretivist is stanch to understanding the social occurrence from the researcher's own perspective and investigating how the world is experienced. This suggests that the author's experience shared in this paper is based on the research process and reflections of the pre-service mathematics teachers. Thus, the author aimed to understand the classroom practices from the pre-service mathematics teachers' perspective and examine their experiences. The construction of meaning during the research process was subjective (Guba & Lincoln, 1994), what and how meanings were embodied in language and action (Schwandt, 1994) of pre-service mathematics teachers was elucidated and each individual pre-service mathematics teacher's unique understanding was valued.

4.2 Participants

Seventy seven third year pre-service mathematics teachers at Makerere University in Uganda were involved in this study. In addition to university mathematics content, these pre-service teachers offer three mathematics teaching methods (MTM) course units in their second and third years of teacher education. In the second semester of their second year, they offer their first ever MTM course titled, 'Teaching and Learning Mathematics' and in each of the two semesters of their third year, they offer one MTM course unit namely 'Curriculum Theory and Contemporary Issues in Mathematics Education' and "Assessment, Evaluation, Resources and Materials Development in Mathematics" respectively. However, in each of the three MTM courses, teaching is via the traditional approaches and further still, in the absence of actual secondary school mathematics content which the teachers are expected to teach after obtaining their Bachelor of Science Education degree. Additionally, at the end of each of their second and third years, the pre-service teachers carry out school practice in secondary schools. Each session of school practice lasts for six weeks, during their semester holidays. The third years were chosen for this study because they had carried out their first school practice and had attended at least one MTM course in the second semester of their second year at the University.

4.3 Data Collection and Analysis

Data in this study were collected through whole-class discussion and pre-service teachers' reflections after the whole class discussion. Reisman, Kavanagh, Monte-Sano, Fogo, McGrew, Cipparone and Simmons (2017) defined whole class discussion as that action in which the teacher and all the students deal with questions or controversies by means of each other's ideas as resources. To them, the intention of such discussions is to "*build collective knowledge and allow students to practice listening, speaking, and engaging in [mathematical]… interpretation*" (p. 279). Thus, in instructionally prolific whole class discussions, the teacher and a varied array of students contribute orally, listen actively, and respond to and learn from others' contributions. Since the author was part of the whole-class discussion, it was video-taped as well as the pre-service teachers' reflections on it, in order to capture all the relevant data. From the video-tape, the teaching and learning process was observed and this was the major evidence for this study. The pre-service mathematics teachers reflected on the whole-class discussion and thereafter discussed the implication of the activity based heuristic approach of teaching and learning to their own teaching, mathematics teacher education and teaching in the 21st century.

The following questions relating to the mathematical learning activities were asked to guide the pre-service mathematics teachers' reflections after the whole-class discussion. Do you think that this session was successful? Why? How do you know? What was the most difficult part of learning in this session? What do you think your strengths would be in teaching this lesson and why? What do you think you would need to develop or improve in your teaching in this lesson and why? What did you learn and how do you know that you learned? What do you know now that you did not know at the start of the session? How do you know? What do you understand now that you did not understand at the start of the session? How do you know? What was not clear to you? Why do you think that it was not clear to you? What are the implications of this approach to the 21st century mathematics teachers? The worksheets of the pre-service mathematics teachers were collected after the session for further evidence. The whole-class discussion lasted for one hour and forty minutes and the reflections took 50 minutes. In order to make meaning of the collected data, analysis was done through narratives of the video observations, which were first written as reflective notes during watching the video. The video on several occasions was paused, rewound or forwarded to capture episodes of interest or clarifications or second opinions by the author. The pre-service mathematics teachers' reflections after the whole class discussion and their implication of this activity based heuristic approach to teaching were discussed.

4.4 Mathematical Activities

This study utilized two mathematical activities namely 1) chains of numbers and 2) people graphs of sequences. The general teaching strategy for teaching the connections between patterns, sequences and graphs was different learning styles namely visual, audio, kinaesthetic (VAK) and the curriculum content was for lower secondary school. The pre-service mathematics teachers were tasked to recognize, describe and represent patterns and relationships in order to develop an understanding and appreciation of the use of a letter to stand for a variable and also to appreciate the relationship between algebraic notation and corresponding graphical representation. As prior knowledge, the pre-service mathematics teachers needed basic number sequences. The intended learning outcomes for the pre-service mathematics teachers at the end of these activities were to: know how to find the algebraic expression for the n^{th} term of a sequence; understand that substituting $n = 1$ gives the first term of the sequence; be able to draw the graph of a linear sequence; appreciate some of the connections which exist between algebraic expressions and graphical representations; and experience a practical activity to support learning.

5. Process of Developing Mathematical Thinking

With reference to the whole-class discussion and reflections of the pre-service mathematics teachers after the whole-class discussion through video observations, in this section the researcher describes the learning of the pre-service mathematics teachers. Video observations and remarks given by the pre-service mathematics teachers are grouped here in line with the two activities that guided this study namely; chains of numbers and people graphs of sequences.

At the beginning of the session, the pre-service mathematics teachers were asked how they remember a cell phone number. Do they see it in their mind's eye? Do they recall it by saying it? Or, do they need the cell phone in front of them to tap it out? According to Pashler, McDaniel, Rohrer and Bjork (2008), many people have a preference for a learning style, either visual, auditory, or kinaesthetic. This means that, given a choice, the learners prefer to learn new information through seeing, hearing or doing in addition to their other senses. For this session, the pre-service mathematics teachers used all the three styles of learning throughout but of course with their other senses, although they just learn better in one way. Some facts that they had to know for the smooth flow of this session were revisited. These included an expression, a number sequence, a term, n^{th} term, a variable, a variable in an algebraic expression, a coefficient and a constant.

Corollary, Linchevski (1995) defines these facts as follows; an expression describes the relationship between numbers while a number sequence is an ordered pattern of numbers. A term is each number in the sequence while the n^{th} term is the defined term in a sequence describing the relationship which exists between the term and its position in the sequence. A variable is a symbol used to describe an unknown number while a variable in an algebraic expression can take different values. A coefficient acts on the variable and a constant stays the same. Using an example of $2n$-1, an elaboration was made where n is a variable, for it takes different values, 2 is a coefficient because it tells you how many of n you have got and 1 is a constant as it does not change.

Activity 1: Chains of Numbers
The following questions were discussed:

What is the n^{th} term for the sequence 2, 4, 6, 8, 10, 12……?

Is the formula for 1, 3, 5, 7 …. $\mathbf{2n+1}$ or $\mathbf{2n-1}$?

Write out the number chains you would get for a class of 77 and differences of 3. Give the formulae for all three chains. Then try other differences and describe the connection between the differences and the formulae.

The topic was not new to the pre-service mathematics teachers, but the general teaching strategy was. At first when the sequence 3, 5, 7, 9……..was written on the chalkboard and pre-service mathematics teachers were told that it was what was going to be explored, they made a lot of noise and sighs in disbelief as they considered this obvious mathematics. They looked mesmerized and one pre-service mathematics teacher asked why we were looking at the cheap secondary school mathematics as compared to the university one that they were currently interacting with. An explanation was given to the effect that because it was part of the mathematics they were expected to teach in secondary schools. When asked to fill in the next two numbers, in unison the pre-service mathematics teachers successfully answered 11 and 13. Then, a volunteer was requested to show the rest of the pre-service mathematics teachers how to get the 100th term. He came and worked it out alone but he struggled and had no clear formulae other than using his head.

Other volunteers were called out to tell or show the class the 151st and 200th terms. Without any discussion or formulae, they wrote on the chalkboard 309 and 407 respectively as their answers. At the moment, the author came in to guide the discussion of activity 1. Seventy seven large cards numbered 1 to 77 were used for this activity. Each of the 77 pre-service mathematics teachers held high up their card, ensuring the number was clearly visible. The pre-service mathematics teachers with the largest even number, 76, and largest odd number, 77, were positioned in two different locations, the extreme ends of the front of the lecture room. While these two pre-service mathematics teachers stayed put, the remaining ones placed their left hand on the right shoulder of the person who had a card numbered two greater than their number. The pre-service mathematics teachers put high up their cards, for both sequences that were formed, that is, the even and odd numbers.

The key questions that were used to develop the pre-service mathematics teachers' understanding on chains of numbers included: What's your number? Can you find someone two more than you? How many chains do you think we will get? There are ten people in your chain, if we had more people what number would the 20th person be? Put your hand up if you think the number 61 would be in your chain. After the pre-service mathematics teachers created the sequences, the whole class described the two sequences which emerged and were written on the chalkboard emphasizing 1st term, 2nd term, etc. Ideally, the author wanted to introduce the vocabulary, term, and to emphasize the relationship between the position and the term. The pre-service mathematics teachers predicted the 10th, 30th, 50th, 100th, n^{th} term/ number and filled in both sequences. Below is what was achieved.

1st	2nd	3rd	4th	5th	10th	30th	50th	100th	n^{th}
2	4	6	8	10	*20*	*60*	*100*	*200*	**$2n$**
1	3	5	7	9	*19*	*59*	*99*	*199*	**$2n$-1**

A4 plain papers and markers were used to enable all the pre-service mathematics teachers to show their ideas.

Having found the pattern as $2n + 1$, the pre-service mathematics teachers were asked if it held for the sequence 1, 3, 5, 7, 9…., placing particular emphasis on 1. They now argued that the algebraic expression was $2n - 1$ after which they were asked to justify their answers. Eventually, they were able to say, in accordance to Greenland *et al*. (2016) that $n = 0$ normally does not hold for people. An important issue discussed here concerned the possible misconceptions from activity 1 and the link between the difference of the sequence and the formula. On the part of the possible misconception, the pre-service mathematics teachers discussed the formula for even numbers, $2n$ and for odd numbers $2n$-1. They figured out that it would be very likely that some learners would express the sequence of odd numbers as $2n$+1 and this would provide a tremendous opportunity to discuss the value of considering the position of each number in the sequence and appreciate the importance of this detail. For the 77 cards, the pre-service mathematics teachers wrote out the below sequences:

2*n* gives the numbers 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32, 34, 36, 38, 40, 42, 44,46, 48, 50, 52, 54, 56, 58, 60, 62, 64, 66, 68, 70, 72, 74, 76 (for n = 1, 2, 3…)

2*n*+1 gives the numbers 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 35, 37, 39, 41, 43, 45, 47, 49, 51, 53, 55, 57, 59, 61, 63, 65, 67, 69, 71, 73, 75, 77 (for n = 1, 2, 3…)

2*n*-1 gives the numbers 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 35, 37, 39, 41, 43, 45, 47, 49, 51, 53, 55, 57, 59, 61, 63, 65, 67, 69, 71, 73, 75, 77 (for n = 1, 2, 3…)

Since the pre-service mathematics teachers wanted the sequence starting with 1, the formula is $2n$-1. They observed that they should never be tempted to rush here as some learners would require additional thinking time and discussion to be convinced that the answer is $2n$-1. They further discussed the formula used to describe multiples of any number that has no constant at the end (Goodman, Li & Tiwari, 2011), for example, $2n$. Such a formula holds true because when one divides the multiple by that number, it has no remainder, or the remainder is zero. The pre-service mathematics teachers flashed back to the two answers, 309 and 407 that were earlier given for the 151st and 200th terms respectively by the two volunteers and confirmed that they were wrong due to failure by the volunteer pre-service mathematics teachers to explain any procedures (Dolev & Even, 2015) they took in addition to working out the answers from their mind.

The activity was repeated, this time fixing in position the pre-service mathematics teachers with the largest three numbers on their cards namely, 75, 76 and 77. The remaining pre-service mathematics teachers placed their left hand on the right shoulder of the person who had a card numbered three greater than their number. The outcome was discussed in a similar way as the 'two greater than'. The pre-service mathematics teachers noticed the relationship between the numbers increasing by 3 in each sequence and the $3n$ in the first part of the n^{th} term. The other sequences were $3n$-1 and $3n$-2. The pre-service mathematics teachers discussed and wrote out all the 'three more than' sequences for a class of 77 people and found the formula for each chain to the effect that:

3*n* gives the numbers 3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36, 39, 42, 45, 48, 51, 54, 57, 60, 63, 66, 69, 72, 75 (for n = 1, 2, 3…)

3*n*-1 gives the numbers 2, 5, 8, 11, 14, 17, 20, 23, 26, 29, 32, 35, 38, 41, 44, 47, 50, 53, 56,59, 62, 65, 68, 71, 74, 77 (for n = 1, 2, 3…)

3*n*-2 gives the numbers 1, 4, 7, 10, 13, 16, 19, 22, 25, 28, 31, 34, 37, 40, 43, 46, 49, 52, 55, 58,61, 64, 67, 70, 73, 76 (for n = 1, 2, 3…)

The pre-service mathematics teachers explored two different sequences that begin 7, 10, 13, 16 … and 1, 4, 7, 10, 13, 16 …. Both these algebraic formulae begin with 3n, but do they finish with the same constant? Although one can see that 7, 10, 13, 16… is a subset of 1, 4, 7, 10, 13, 16…, both these sequences have different first terms and although numbers are similar but their positions are different. It turns out that the algebraic representation of 7, 10, 13, 16 …is $3n$+4 while that of 1, 4, 7, 10, 13, 16 …is $3n$-2. Regarding the link between the difference of the sequence and the formula, for a difference of 3, the sequences as earlier stated are $3n$, $3n$-1 and $3n$-2. The algebraic notation always includes $3n$ as the numbers in consecutive terms increase by 3. Also recall that the term $2n$ was in the sequences with a difference of 2.

The pre-service mathematics teachers then created 12 small groups of six and one of five, each to decide how to organize the activity to produce sequences with a difference of 4. They were able to predict the formulae namely $4n$, $4n$-1, $4n$-2 and $4n$-3. One of the groups was chosen to give instructions so that the whole class made sequences with a difference of 4. The pre-service mathematics teachers predicted the formulae for the sequences with a difference of 4 as $4n$, $4n$-1, $4n$-2 and $4n$-3. They then wrote out the sequences for a class of 77:

4*n* gives the numbers 4, 8, 12, 16, 20, 24, 28, 32, 36, 40, 44, 48, 52, 56, 60, 64, 68, 72,76 (for n = 1, 2, 3…)

4*n*-1 gives the numbers 3, 7, 11, 15, 19, 23, 27, 31, 35, 39, 43, 47, 51, 55, 59, 63, 67, 71, 75 (for n = 1, 2, 3…)

4*n*-2 gives the numbers 2, 6, 10, 14, 18, 22, 26, 30, 34, 38, 42, 46, 50, 54, 58, 62, 66, 70, 74 (for n = 1, 2, 3…)

4*n*-3 gives the numbers 1, 5, 9, 13, 17, 21, 25, 29, 33, 37, 41, 45, 49, 53, 57, 61, 65, 69, 73, 77 (for n = 1, 2, 3…)

They then concluded that the difference in two consecutive numbers in the sequence gives the number of the sequences that emerge:

- If the difference is 2, then the number of sequences is 2 namely **2*n*** and **2*n*-1**
- If the difference is 3, then the number of sequences is 3 namely **3*n***, **3*n*-1** and **3*n*-2**
- If the difference is 4, then the number of sequences is 4 namely **4*n***, **4*n*-1**, **4*n*-2** and **4*n*-3**

This general teaching strategy applied for teaching patterns, sequences and graphs because each pre-service mathematics teacher preferred a learning style (Pashler *et al.*, 2008) that was different from others. Using different learning styles helped the author to strike for a balance of instructional methods so that all the pre-service mathematics teachers met a variety of experiences. Although the author's learning style preference is actually visually doing, trying the cell phone test gave the pre-service mathematics teachers and the author an idea. The reason it is good to know is that teachers tend to teach in the way that they themselves prefer to be taught (Cox, 2014). So if one prefers learning through listening they are likely to lecture their learners, if they prefer learning by using pictures and diagrams they are likely to demonstrate through illustration, and if they prefer learning by doing they are likely to set up activities where their learners have hands-on experiences.

Activity 2: People Graphs of Sequences

The 77 large cards used in activity 1 were used along with both A4 and graph papers for this activity. Numbers were written on A4 pages and the y-axis comprised numbers 0 to 10 while the x-axis 0 to 5, as illustrated in Figure 2. Given a large class size of 77 pre-service mathematics teachers, the numbers written on A4 pages were assembled in an open field and pre-service mathematics teachers were organized to produce Cartesian x and y axes on the ground. Meanwhile, the pre-service mathematics teachers helped to make the axes and to begin with, the first part of Activity 1 - chains of numbers was repeated in order to form the odd and even sequences and to recap the sequences produced as well as the language used, that is, position and term/ number. The Cartesian plane for the x-axis and the y-axis were organized on the ground and pre-service mathematics teachers formed chains, thus recalling their learning. The first pre-service mathematics teacher stood on (1, 2), the second, third, fourth and fifth pre-service mathematics teachers stood on (2, 4), (3, 6), (4, 8) and (5, 10) respectively. The x-co-ordinate was position in the sequence and y-co-ordinate was the term value.

The pre-service mathematics teachers who formed the chain of even numbers 2, 4, 6… formed a graph and were asked if their graph went through the origin and further to name the shape of the graph, which was linear. They then looked for the pattern (Stern, Kalech & Felner, 2012) for consecutive people in their graph and to that end, five consecutive people in the graph were asked to call out their co-ordinates which were (1, 2), (2, 4), (3, 6), (4, 8) and (5, 10). Further, the pre-service mathematics teachers were asked for the pattern they observed and where the graph would cross the x-axis and why. This was a straight line passing through the origin and it passed through the origin because for $n = 0$ in the sequence $2n$, $2n = 0$.

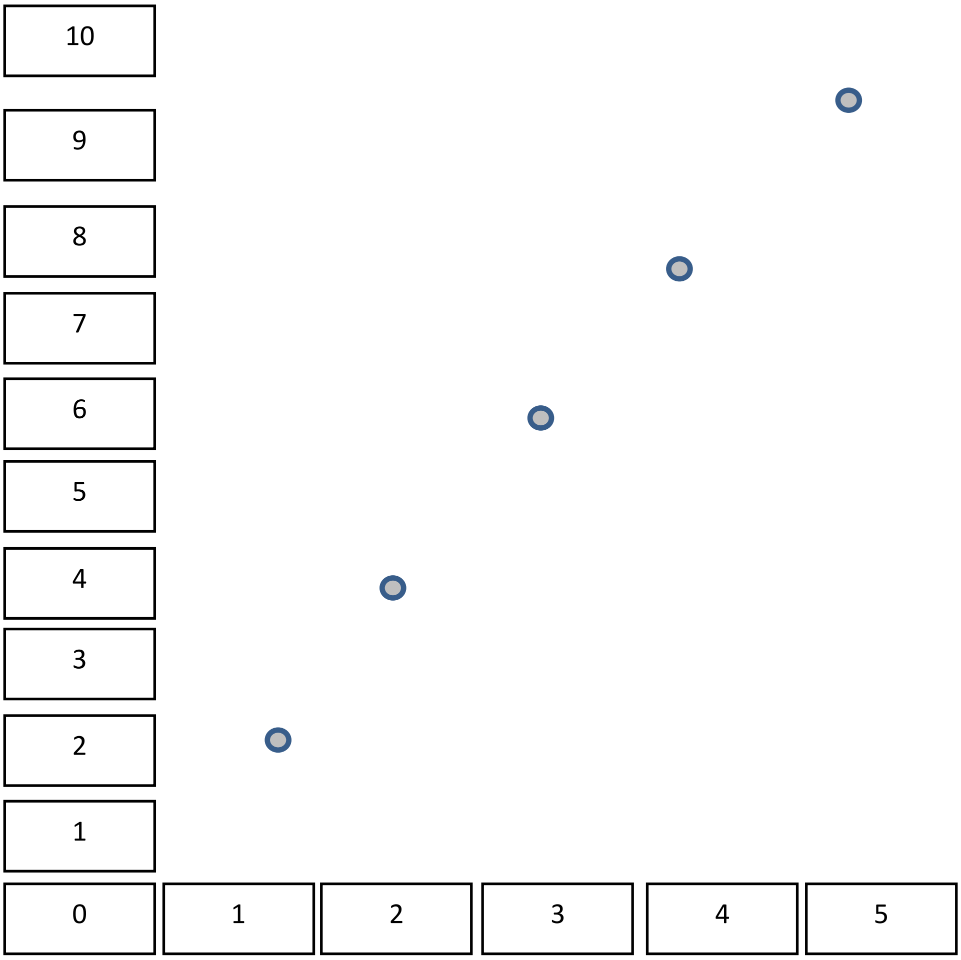

Figure 2: Cartesian plane on the Floor

The pre-service mathematics teachers repeated for the odd numbers $2n$-1 and the ones in the odd number chain now formed a graph. Pre-service mathematics teachers described, discussed, explored and told the author about the graph for odd numbers. They compared the graphs for even and odd numbers to establish similarities and differences. To introduce the vocabulary y-intercept, the pre-service mathematics teachers were asked to join the points by a straight line and to locate where this line cuts the y-axis. They used graph papers to show the sequences and described where the graphs cut the y-axis. First, they worked individually to draw the graphs and then discussed in pairs, describing the graphs using the formal vocabulary they had learned.

In this activity of people graphs of sequences, the pre-service mathematics teachers were not too concerned with the accuracy because people graphs only give a rough visual representation but however, they certainly observed the required relationships. They were clear that each card stood on the Cartesian plane in a location that represented its position in the sequence and the value of the term. The graph was an obvious straight line (see Figure 2 again). The pre-service mathematics teachers developed the idea that it was necessary to move one square horizontally and two squares vertically to move between consecutive points. The graph cuts through the origin since n = 0 results in $2n$ = 0. It was important to appreciate that in a real situation such as people or counters n = 0 is not valid (Goodman *et al.*, 2011), the reason $2n$+1 could not hold true for 1, 3, 5, 7…. For odd numbers, the graph was still a straight line, however the y-intercept was -1. The pre-service mathematics teachers also explained to each other why one cannot produce one people graph to represent sequences expressed as $3n$, $3n$+1 and $3n$-2 if only one person represented each natural number. They explored if $3n$, $3n$+2 and $3n$-1 would work and described algebraically the sequence of a graph with coordinates (1, 5), (2, 8), (3, 11), (4, 14) and that cuts the y-axis at +2 which was found to be 3n+2.

The pre-service mathematics teachers produced arrangements of counters to represent this sequence and discovered further that when the lines actually intercepted the y-axis at +1 instead of +2, then the y coordinate values would drop by 1 unit for each pair of coordinates. They looked at the pattern in the parallel graphs that emerged. By using mathematical activities to discuss (Celik, 2018), the pre-service mathematics teachers were able to help each other to learn and understand the secondary school mathematics concepts. Since this method of approach, especially connecting patterns, sequences and graphs was new to the pre-service mathematics teachers, it was the only way they discussed for teaching this topic. The session begun with what the pre-service mathematics teachers knew about sequences before the activities could be introduced. It was expected that the pre-service mathematics teachers would be confident to be introduced to the idea of the steepness of the graph and explained to that it is called the gradient as the gradient has the same value as the difference of the linear sequence. They looked at how the steepness and the gradient of the line changed when represented on a graph if a formula had a constant at the end, and the assumptions they could make about the sequence of numbers if a graph cuts through the origin.

6. Results and Discussion

The conversational framework (Laurillard, 2002) indicates that activity based teaching and learning culminates into a conversation between the teacher and the students and constitutes teacher's conception, student's conception, student's actions and teacher's constructed environment. In this section, the author presents the findings from the whole-class discussion and reflections after the whole-class discussion and have been presented according to the constructs of the conversation framework.

6.1 Teacher's Conception

Laurillard (2002) maintains that teacher's conception involves students' conception after the teacher shares their ideas, students' re-description after the teacher's conceptions and reflection on learners' actions to modify descriptions. Once the students share their re-described ideas with the teacher, they are free to re-describe their ideas, too, depending on their re-described ideas. This keeps the conversation of learning spiral. But this means that the teacher has the responsibility of finding out what the learners come to class with through sharing ideas. In this study, the teacher used students' prior knowledge of patterns to build up to sequences and graphs. This was a very good ground for developing students' mathematics self-concept which later translated to mathematical thinking. This finding is in agreement with Delima *et al*. (2018) who found that students with positive mathematics self-concept had good mathematical thinking abilities because they focused on specializing, generalizing, conjecturing and convincing. If this were the way pre-service mathematics teachers were trained to teach (Cox, 2014) secondary school mathematics, there would be no such issues like poor and theoretical teaching methods and poor performance in mathematics among learners in Uganda, for this approach aids learners to think and reason logically (Hudson, Henderson & Hudson, 2015). Although it was a struggle to convince the pre-service mathematics teachers about the relevance of this session at the beginning, especially dealing with secondary school mathematics that they do not interact with at the university, the session supported them in developing a deep understanding of the secondary school mathematics and exposed them to developing more effective ways of teaching. At the end of the session, the pre-service mathematics teachers wondered why they had not learned like this all along. They were excited about the session and consequently, thorough understanding must have taken place.

6.2 Students' Conception

According to Laurillard (2002), students' conception involves the teacher's ideas and their re-description after sharing with the students and reflection on concept in light of experience. From reflections, pre-service mathematics teachers noted that they should make sure that they provide some visual clues and use diagrams and pictures during their teaching, which is in agreement with Fuson (2019). For example, during this session, pre-service mathematics teachers were asked to draw their own representations of the mathematics that we were doing together and further close their eyes and answer questions about the graph we had drawn, one that they could visualise. Such conception was very helpful in developing mathematical thinking. Several scholars, for example Kargar *et al*. (2010), Nepal (2016) and Yorulmaz et al. (2017) have linked the development of mathematical thinking to benefits such as positive mathematics attitude, high achievement and low mathematics anxiety among students. While Kargar *et al*. (2010) found a high correlation between mathematical thinking and attitude, Nepal (2016) found mathematical thinking to relate with achievement and that there were no gender differences in mathematical thinking among students who achieved highly in mathematics. Meanwhile, Yorulmaz et al. (2017) found high mathematical thinking scores to relate with low anxiety scores, thus, there was a negative significant correlation mathematical thinking and anxiety.

6.3 Students' Actions

Regarding students' actions, Laurillard (2002) maintains that these incorporate goals set by teacher, feedback from the teacher's constructed environment and adaptation of actions in light of the goal(s) and feedback. Pre-service mathematics teachers had to do something with the cards, A4 papers, graph papers by moving their own bodies, thus, realizing that there are lots of everyday objects that can be used to model mathematics, for example buttons, bottle tops, stones, strings, straws among many others. As compared to previous sessions, these activities engaged the pre-service mathematics teachers in learning visually, by listening (audio) and by moving around (kinaesthetic). The pre-service mathematics teachers felt engaged and provoked to think when they learned in different ways (Pashler *et al*., 2008). They preferred visually doing style of learning more than the other styles and discussed the pros and cons of teaching so that learners could get the opportunity to learn in different ways, especially audio and kinaesthetic. The strongest point in the pre-service mathematics teachers' mathematics learning was the ability to work out the sequences of the differences of 3 and 4 on their own. This was a clear indicator that they had understood from the activity of sequences of 2. On observing what the pre-service mathematics teachers recorded, it was noticed that some of them had gone beyond the difference of 4 to those of 5 and 6.

Nonetheless, some pre-service mathematics teachers had attempted a few times before they identified the sequences. However, the most difficult part of the teaching was moving from basic numerical work into algebraic representation and manipulation. These activities used concrete practical situations to gently move into the abstract as advocated by Vygotsky (1978). These tasks allowed pre-service mathematics teachers the chance to develop a deep understanding and truly appreciated the benefits of algebraic notation and procedures rather than the previously learnt techniques that they easily forget, thus becoming meaningless. Although the link between a sequence of numbers, the n^{th} term, the graph produced and the structure of the shapes from which the sequence emerges is fundamental, this was the first time the pre-service mathematics teachers were establishing it. Such relationships enabled them to appreciate that learning mathematical concepts is a process, as Bakker, Smit and Wegerif (2015) observe, which intertwines and not lots of individual concepts which do not relate to each other. Thus, learning mathematics is not a linear process. Singh *et al*. (2018), Pokhrel (2018) and Delima (2017) found problem solving activities to boost mathematical thinking. Li *et al.* (2019) stressed that mathematical thinking emphasizes the process of mathematical methods application in problem solving, as was the case in this study, and also focuses on making the concept more understandable.

6.4 Teacher's Constructed Environment

The indicators of teacher's constructed environment according to Laurillard include adaptation of task goal in light of students' description, students' actions and students' modified actions. In this regard, the author endeavored to give good verbal explanations, encouraged pre-service mathematics teachers to talk about what they were doing, asked them to explain their ideas (Staples, Bartlo & Thanheiser, 2012) to the whole class and to each other because one cannot do mathematics if they do not talk mathematics. For the low achievers, time was taken to explain how to transit from sequences to algebraic expressions and graphical representations. For even numbers 2, 4, 6, 8.... a pattern was built and for the odd numbers different pairs of the sequence of 1, 3, 5, 7..., were considered one at a time. While still struggling with the lower achievers, the higher achievers worked with 'three more than' and 'four more than'. Thus, it's the students' actions and modified actions that give way to teachers to construct a learning environment.

7. Implication for Pre-Service Mathematics Teachers

The activity based heuristic approach is helpful in mathematics learning in a meaningful yet an enjoyable way. The students are able to visualize mathematics in their context and setting in the learning process. Thus, there is need for an exigent change in the teaching approaches during pre-service mathematics teacher training at Makerere University to suit the professional practice of 21st century mathematics teachers. For example, Uganda's secondary school lasts for six years before learners join institutions of higher learning, where they spend three years, say, to train as secondary school teachers. During each of their second and third years at the teacher training course, the pre-service mathematics teachers at Makerere University accomplish school practice, each turn taking six weeks. Usually as student teachers, they are assigned lower secondary level classes, that is, Seniors One and Two where mathematics content is believed to be manageable at least by their own measure. Meanwhile, also anecdotal evidence shows that even when not allocated these lower level classes, the pre-service mathematics teachers request for them owing to their fear and lack of confidence (Karigi & Tumuti, 2015) to teach higher levels. This is a similar situation as that indicated in Zhu *et al*.'s (2018) study where the non-expert teachers could not rely on prior teaching experiences because they did not know how to. In this particular context, the pre-service mathematics teachers cannot translate the traditional replicative mathematics teaching methods trained at the university into the actual teaching practice, for they do not know how to.

The fear and lack of confidence is not surprising because in essence, the last time the pre-service mathematics teachers interacted with Seniors One or Two mathematics content was when they were learners at that level and in reality, about eight years back. This is where the missing link in mathematics teacher training is! The eight year pedagogical and content gap compromises the competency of the pre-service mathematics teachers, leading them to actually recollect their experiences as learners and use these as either pre-service and/ or in-service mathematics teachers to teach, a situation that might have prompted Cox (2014) to ask if teachers taught as they were taught. Further, since the pedagogical training the pre-service mathematics teachers do obtain at Makerere University is in the absence of secondary school mathematics content, there is urgent need to incorporate strands of secondary school mathematics in the mathematics teacher training curriculum, in addition to the advanced mathematics content they study. I am acutely aware that this advanced mathematics content is much needed to make the pre-service mathematics teachers well prepared and knowledgeable in their respective subject matter, however, the compliance of the curriculum is important. Interacting with the secondary school mathematics content that the pre-service mathematics teachers are expected to teach will translate traditional teaching approaches, where the mathematics teachers solve problems for themselves at the chalkboard because they do not know why they do what they do (Eko, Prabawanto, & Jupri, 2018) but simply were taught that way, to active learning.

It is important to know that explaining something to someone else is a real test of how well one understands it and learners will need a lot of help and strong modelling from teachers if they are going to move away from 'this is what you do' to 'I can explain what this means'. Unless Makerere University takes strong strides in reconsidering the current mathematics teacher education pedagogical approaches, it shall continue to be challenged with training 21st century mathematics teachers who can help learners to develop mathematical thinking, creativity, imagination and skills that apply to practical activities such as communication, feedback, organizing, planning, problem-solving and persuasion. Yet, according to Hudson *et al*. (2014), mathematical thinking is essential for teaching and learning mathematics. It is not only the foundation for learners' development and a basis of their sustainable development in mathematics, but also helps them to be creative thinkers than being doers of mathematics problems by way of working with activities thus, enhancing learning and understanding in mathematics.

8. Conclusion

This study intended to provide the third year pre-service mathematics teachers at Makerere University with an experience of using actual secondary school mathematics in making connections between mathematical topics using mathematical activities. Based on the activities used during the whole-class discussion session of this study, it was observed that learning mathematics through activities helps learners to develop mathematical thinking which is the foundation for their development and a basis for their sustainable development in mathematics. The learners are engaged in the learning process and besides, develop several other competences like communication, creativity and innovation, presentation and working in a team. Unless Makerere University considers revisiting the pedagogical training of pre-service mathematics teachers, there are high chances that mathematics achievement by learners will continue to be at stake, thus producing citizens who cannot compete as 21st century global citizens in terms of critical thinking, collaboration, curiosity, confidence, creativity and communication.

8.1 Limitations of the Study

Although the study found that the activity based heuristic approach was helpful in developing the mathematical thinking of pre-service mathematics teachers at Makerere University, this cannot be generalized to all pre-service mathematics teachers in Uganda. Further, the second year pre-service mathematics teachers could also have participated in the study. Also, since the author is one of the facilitators of the third year pre-service mathematics teachers, she played an insider role, having details of how critical the study problem was, without necessarily pointing to wide literature but rather anecdotal evidence.

9. Recommendations

Makerere University is one of the nine government universities in Uganda. Given the importance of the major finding of this study, which could be relevant to all pre-service mathematics teachers in Uganda, similar studies need to be done in the rest of the eight government universities. There is also urgent need for comparative studies both within the local context, across universities, and international context, across continents. Such studies would be helpful in ascertaining similarities and differences in content and context in mathematics teacher education.

Declarations

The author declares that there are no competing interests.

Acknowledgements

Much appreciation goes to Margaret Babirye Lwebuga (MBL) for the scholarship that enabled the researcher to attend the Mathematical Thinking Course by the African Institute for Mathematical Sciences Secondary Enrichment Centre (AIMSSEC) at Makerere University. This article is a product of the course and the activities used herein were drawn from the AIMING HIGH Teacher Network which is published under a Creative Commons copyright license and AIMSSEC.

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