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# ENGAGING DIGITAL TECHNOLOGIES TO EXPLORE SOLUTIONS OF EQUATIONS INVOLVING MODULUS FUNCTIONS

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**Abstract:**
This study explored how dynamic mathematics software package called GeoGebra contributed to participants' learning and understanding of equations involving modulus functions. The research followed a case study approach with a small group of six participants in a Sixth Form classroom in England. The research focused on participants' experiences as they used technology to support their understanding of the concept: *modulus functions*. It highlights how participants used GeoGebra to correct some misconceptions about equations involving modulus functions and also investigate the source of some spurious answers obtained when using algebraic methods to solve the equations. The focus of the study was on how participants utilised GeoGebra to address misconceptions and perceptions about modulus functions. The main research questions that guided this study focused on how participants used GeoGebra to support their understanding of modulus functions and how GeoGebra related and contributed towards their whole learning experiences. The study found that GeoGebra provided a medium of visualisation that linked abstract aspects of modulus functions with graphical illustrations. Conclusion: Working with GeoGebra extended participants' understanding of modulus functions.

**Keywords:** GeoGebra; multiple representation; visualisation; modulus functions

## 1. Introduction

The new A Level Curriculum (first taught in September 2017) states: *"The use of technology, in particular mathematical and statistical graphing tools and spreadsheets, must permeate the study of AS and A Level Mathematics".* This study investigated the use of a multiple representational software GeoGebra, to explore solutions of equations involving modulus functions. The study is a follow up to a research study that focused on participants' experiences of using GeoGebra to understand the concept of modulus

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functions. The study highlights some of the common errors students make when solving equations involving modulus functions. This follow up study continued to use GeoGebra to provide a multiple representation platform to aide in the understanding of modulus functions and investigate solutions of equations that involve modulus functions.

Learners experience problems in solving equations that involve modulus functions. The MEI examiners' reports highlights some of the common errors observed while candidates attempted to solve $|2x -1| = |x|$. The June 2011 examiners' report notes:

*Candidates attempted this by considering $\pm(2x - 1) = \pm x$, some thinking that this led to four different possibilities, and indeed finding more than two solutions by faulty algebra. A few squared both sides, found the correct quadratic, and solved it by factorising or formula. Some had no idea how to start and tried to manipulate the equations with modulus signs, often ending with the answer $|x| = 1$ or $x = \pm 1$. Others thought the modulus signs implied inequalities and either replaced the equality sign with the inequality sign or introduced the inequality sign in their answers* (Examiners Report MEI Core 3 June 2011, Question 1).

Gono (2016) observed some spurious answers obtained from faulty mathematical calculations of $|2x - 1| = 3x$, when learners solved the equation algebraically. By considering $2x - 1 = 3x$ and $-2x + 1 = 3x$, the participants came up with two solutions instead of one. The Examiners' report and Gono's (2016) observations highlight problems that participants face when solving equations involving modulus functions.

GeoGebra was chosen because of its interface that displays mathematical concepts in multiple representational formats on the same screen. The use of multiple representations has been strongly connected with the complex process of learning in mathematics, and more particularly, with seeking participants' better understanding of important mathematical concepts. GeoGebra provided a medium for visualisation that linked the abstract aspects of modulus function equations with graphical illustrations.

### 1.1 The role of multiple representations on teaching and learning

Socio-cultural discourses have increasingly recognised multiple representations as central to the appropriation of knowledge through representational activity. The discourse on the nature and role of representational environments is well established, with several key texts devoted solely to this theme (Confrey, 1990; Janvier 1987, Kaput 1989).

Research in cognitive science and mathematics education has focused on the important role played by multiple representations in mathematics learning (Mehdiyev, 2009; Ruthven et al, 2008; Yerushalmy & Schwartz, 1999; Dufour-Janvier et al, 1987; Kaput 1989). Yerushalmy & Schwartz (1999), claim that multiple representations allow learners to use a rich set of both symbolic and graphical representations, hence building a richer and deeper understanding of mathematics concepts. Panasuk (2010) cites research studies (Ainsworth et al 2002, Lowrie, 2001 and Swafford & Langrall, 2000) that have focused on multiple representations and their subsequent impact on learning mathematical concepts. Stenning, Cox & Oberlander (1995) cited in Ainsworth et al

(1999) gave learners graphical calculators and found that high performing participants benefited from using graphical calculators while lower performing participants preferred textual instructions. Ozgun-Koca (2001) argues that the use of multiple representations has unavoidable contributions on meaningful algebra learning. Bayazit and Aksoy (2007) concur that representing the same concept in two different ways on the same screen promotes learners' depth of understanding and development of knowledge across the representations.

Cordova & Lepper (1996), Cox et al. (2003) and Dikovic (2009) found that multiple representations positively affect pupils' attitudes and understanding of mathematical concepts. Dufour-Janvier et al (1987), investigating the accessibility of representations concluded that multiple representations mitigate learning difficulties by making mathematics more attractive and interesting. Duncan (2010) found that multiple representations stimulate investigations and help participants develop their own learning skills. Studies by Ozgun-Koca (2001) and Pitts (2003) provide evidence about the effectiveness of multiple representations based instruction in college algebra course. They found that the use of multiple representations enabled participants to establish connections between varieties of representational modes.

Schoenfeld, Smith and Arcavi (1993) cited in Ainsworth (1999) examined one participant's understanding of mathematical functions using a multi-representational environment that exploited both algebraic and graphical representations to support learning. The participant's increasingly successful performance led the researchers to conclude that she had mastered fundamental components of the learning domain by exploiting multiple representations involving algebra and graphs. Findings from research studies conducted by Orton (1983) and Tall (1985) indicate that the use of multiple representations is advantageous in promoting conceptual understanding of graphs of derivatives.

However, there are some noted weaknesses or disadvantages of using multiple representations in mathematics teaching and learning. Advocates of a constructivist approach to education argue that dynamically linking representations through use of technology leaves a learner too passive in the process (Ainsworth 1999). Ainsworth argues that such dynamic links discourage reflection on the nature of the transformation, leading to failure by the learner to construct the required understanding. Participants can see things dynamically changing but might not understand how they are changing.

## 2. Theoretical Frameworks

This study follows an Interpretive Phenomenological Analysis (IPA) approach to gain insight into participants' experiences while using technology to solve equations involving modulus functions. Interpretive phenomenological analysis is an approach to qualitative enquiry (Smith et al 2012). IPA is usually used to study small numbers of participants aiming to reveal the experience of each individual. It is concerned with the detailed examination of lived experience. The rationale for using IPA in this research was to explore and describe how learners experienced technology in the process of solving equations that involved modulus functions. With the classroom setting as the direct source of data an IPA approach was appropriate to capture real-time experiences as learners were at work.

## 3. Methods

The research design selected for this study incorporated IPA theories within a socio-constructivists paradigm, in an attempt to explore how participants interacted with technology in solving equations involving modulus functions. Focusing on a small group of participants, working with GeoGebra to understand a specific phenomenon suited a case study approach. Data was collected through screencast video recordings and compiled field notes. The IPA approach allowed for an in-depth understanding of the experiences and perceptions of participants as they used GeoGebra to explore equations involving modulus functions and how they developed an understanding of the concept.

Data was drawn from multiple sources such as observations, interviews and video materials. To understand how participants used GeoGebra to enhance their understanding of solutions of modulus functions, data was collected from participants' screencast and audio recordings on their laptops. Pirie (1996) cited in Powell et al. (2003) observes that videotaping a classroom phenomenon is likely to be the least intrusive, yet most inclusive way to study the phenomenon. The screencast recording software (JING) captured all on-screen activities highlighting a variety of approaches employed by participants.

### 3.1 Data Analysis

On-screen activities and conversations were recorded during each activity and an IPA approach was used to analyse the data (Smith et al 2012). Data analysis started with organisation and description, modelled around an interpretive phenomenological analysis framework (Smith et al, 2012). Initially, the activities in the learning arrangements served as the units of analysis.

## 4. Activities

### 4.1 Solving equations involving modulus functions

The first method investigated was the use of the mathematical definition of an absolute value function. An absolute value $|x|$ of a real number x is the non-negative value of x. The absolute value $|x| = x$ for positive x and $|-x| = x$ for a negative x. This activity highlights the success and failure when this definition is applied in solving equations that involve modulus functions, as depicted in participants' work. The activities in this section focused on participant's experiences as they solved equations involving modulus functions algebraically and graphically. Participants completed several tasks but this paper highlights a selected few cases that provide rich data relevant to the focus of this discussion.

**Case 1:** ***Solving*** $|x + 2| = 2x + 1$

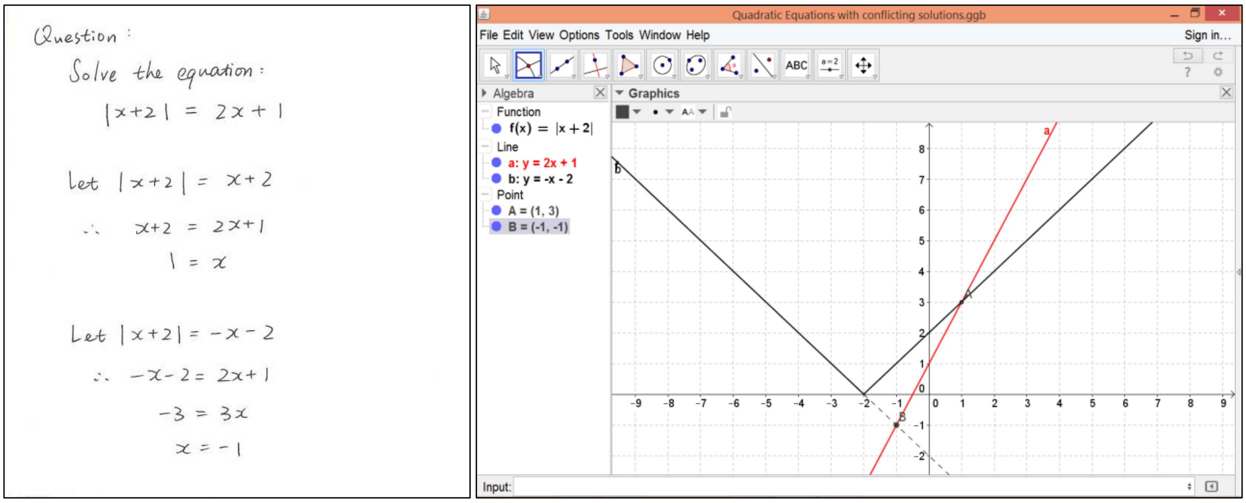


**Figure 1.1:** Graphs of f(x) = $|x + 2|$ *and* $y = 2x + 1$

The participant used positive and negative properties of an absolute value to split the equation into two cases (See handwritten insert above) and got two solutions x = 1 and x = -1.The participant did not see anything mathematically wrong with the calculation. However, when asked to check which of the two solutions was valid, substituting x = -1 into the equation $|-1+2|=2(-1)+1$ gave $|1| = -1$. This contradicted the definition of an absolute value. Graphically representing the same functions in GeoGebra (Figure 1.1) clearly illustrated that there is only one point of intersect. The line y = 2x + 1 intersects with f(x) = - x - 2 outside the range of f(x) = $|x + 2|$ (see Figure 1.1), thus rendering the solution x = -1 invalid. The only valid solution for $|x + 2| = 2x + 1$ is x = 1.

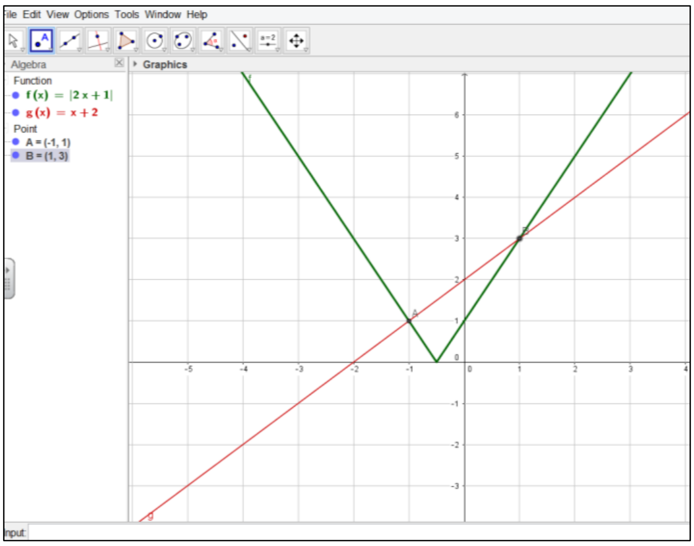


**Figure 1.2:** Graph of y = $|2x + 1|$ and y = x + 2

The same algebraic method is however successfully applied to solve a similar equation |2x + 1| = x + 2. Applying exactly the same algebraic steps as in Case 1 above yields the solutions x = 1 and x = -1 which are both valid.

Graphical illustrations show the difference between Figure 1.1 and 1.2, which could not be identified by using the mathematical definition of modulus functions.

**Case 2:** f(x) = |*x* - 2| *and g(x)* = |*4x* - 3|. *Find the coordinates of the points where f(x) and g(x) intersect.*

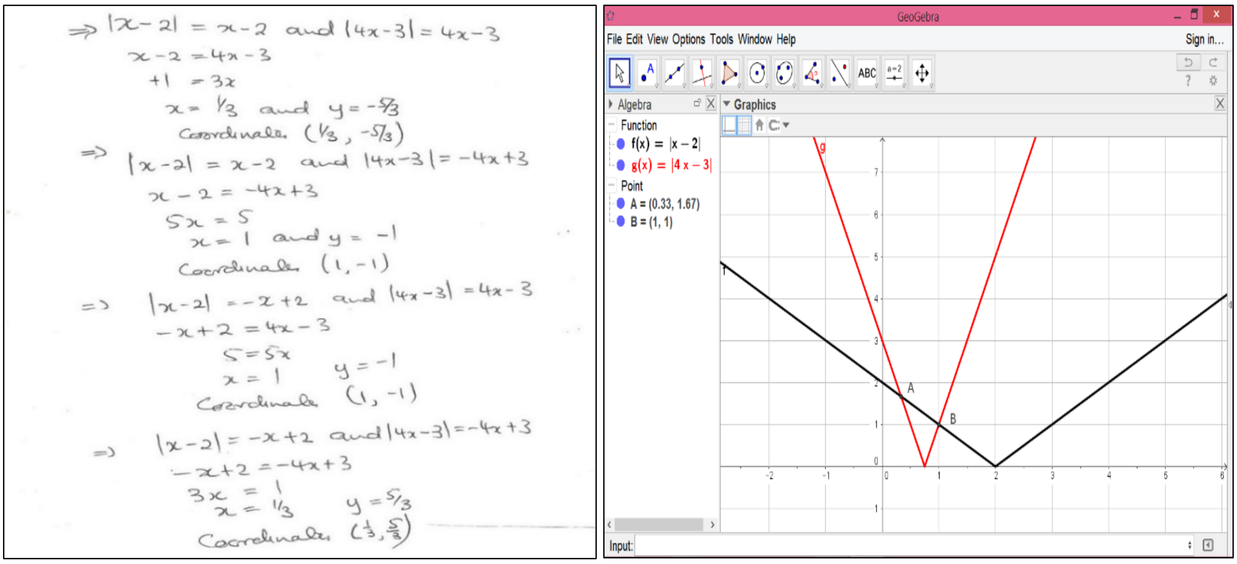


**Figure 1.3:** Graphs of f(x) and g(x) displayed on the same screen

To find the points of intersection, some participants solved the equation |x – 2| = |4x – 3|. Similar to the first activity above, participant used the definition of absolute value functions and considered cases where |x–2|= x–2; |x–2| =-x+2 and the same for |4x – 3| (see Case 2). The participant's work copied above clearly identifies four solutions (1/3, -4/3), (1, -1), (1, 1) and (1/3, 4/3). When probed further, the participant was of the opinion that there were four solutions to this equation, a view that was changed when the functions were presented on GeoGebra.

When the functions f(x) = |x – 2| and g(x) = |4x – 3| were entered in GeoGebra (Figure 1.3), the graphical representations clearly showed that there are only two points of intersection at A (1/3, 4/3) and B (1, 1). This prompted the participant to investigate the disparity between solutions from the algebraic calculations and from graphical representations. Each line was drawn representing all definitions used by the participant in the algebraic calculations i.e. y = 4x – 3; y = -4x + 3; y = |x – 2| and y = -x + 2 (See Figure 1.4). The graphical representations clearly show that the functions f(x) and g(x) are greater than zero for all values of x. Points C and D lie outside the range, hence not solutions to the equation |x – 2| = |4x – 3|.

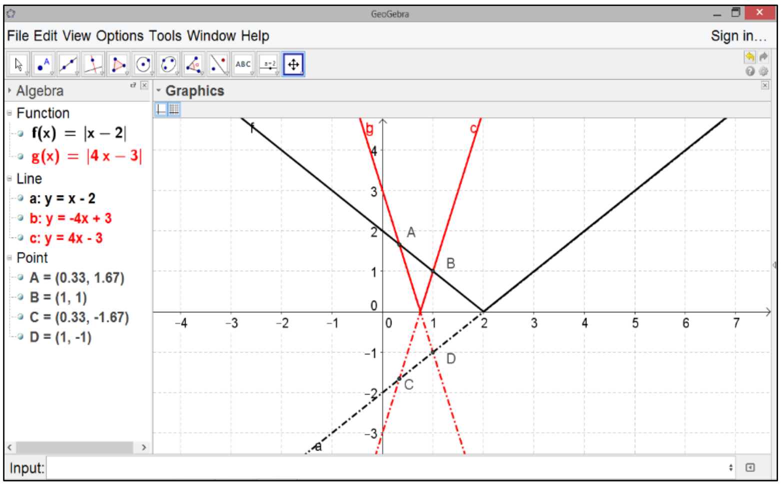


**Figure 1.4:** The graphs of f(x) = ǀx – 2ǀ and g(x) = ǀ4x – 3ǀ

The graphical representations in Figure 1.4 brings to fore the aspects of domain and range of modulus functions that are important to consider when solving equations involving modulus functions. From the graphical representations, participants developed a broader understanding of the mathematical definitions of modulus functions. The graphical representations in GeoGebra shows that the modulus function f(x) = ǀx – 2ǀ is also defined mathematically as

$$f(x) = \begin{cases} x - 2 \text{ for } x \geq 2 \\ -x + 2 \text{ for } x < 2 \end{cases}.$$

This mathematical definition clearly shows that the domain is for all values of x where x Ɛ R. The graphical representation also shows that the range of f(x) ≥ 0. The function g(x) = ǀ4x – 3ǀ is represented by

$$g(x) = \begin{cases} 4x - 3 \text{ for } x > \frac{3}{4} \\ -4x + 3 \text{ for } x < \frac{3}{4} \end{cases}, \text{ range } g(x) \geq 0.$$

The use of GeoGebra to facilitate the visualisation of the abstract concept of modulus functions enhanced the learner's overall understanding of the concept of modulus functions. Bayazit and Aksoy (2007) argue that representing the same concept in two different ways promotes participant's depth of understanding and development of the knowledge.

**Case 3:** Solve the equation **|x| = |2x + 1|**

Another algebraic method of solving equations involving modulus functions emerged in this activity. The participant used the mathematical definition of modulus functions (that is $|x| = \sqrt{(x^2)}$) to solve the equation **|x| = |2x + 1|** and correctly obtained correct solutions.

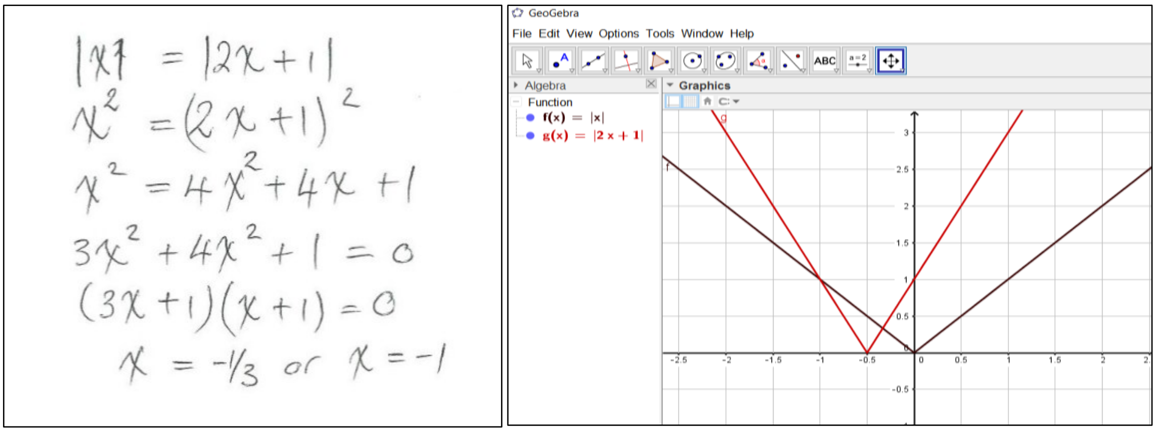


**Figure 1.5:** Illustration of |x| = |2x + 1|

**Case 4:** Solve the equation |x – 2| = -3.

However applying the same definition to solve |x – 2| = -3, creates a false impression that the graphs of f(x)=|x – 2| and g(x) = -3 intersect at two distinct points.

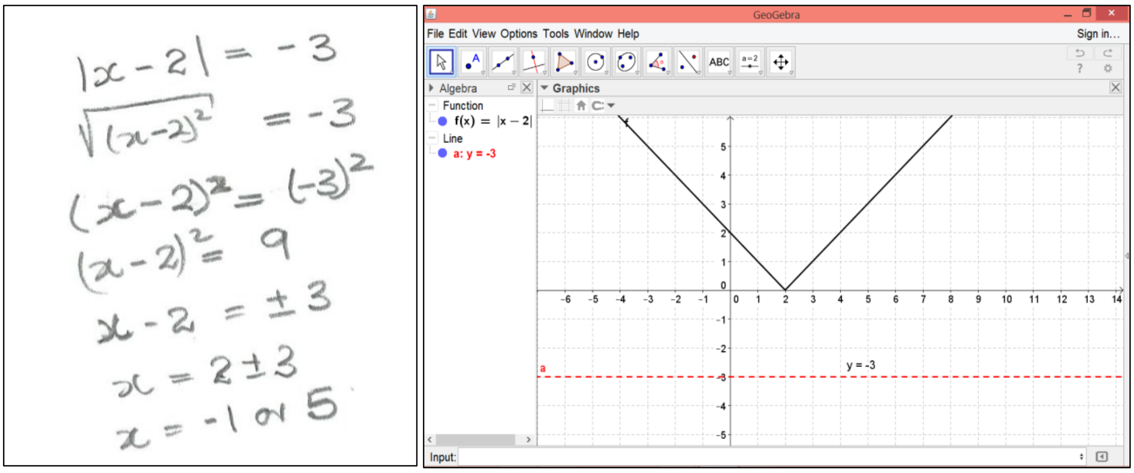


**Figure 1.6:** The graph of f(x) = |x – 2| and y = -3.

A visual display of these abstract functions (Figure 1.6) clearly outlines the limitations of the method of squaring when solving equations involving modulus functions. Figure 4.6 displays the graphical representations of f(x) = |x – 3| and the line y = -3. The graphical representation clearly shows that the two graphs do not intersect. Graphical representations and discussion with other participants highlighted the fact that an absolute value function cannot be equated to a negative constant.

This study also observed that using graphs to solve equations involving modulus functions was mostly a method of last resort. Graphical representations were only used to check solutions from the algebraic method. It was not a method of first choice for most participant.

## 5. Findings

It is so often that participants treat equations that involve modulus functions like ordinary equations without considering the actual mathematical definition of modulus functions.

Using technology to solve equations involving modulus functions allowed participants to pay attention to detail, while at the same time answering the question "Why?" Throughout the investigations, participants looked at various methods of solving equations that involved modulus functions and weighed the merits and demerits of each method. They highlighted the successes and failure of each algebraic method.

Investigations using graphical representations on GeoGebra allowed participants to come up with generalised results, which they noted down:

- Equations of the form $|x + a| = x + b$, have no roots if $a > b$, but will have only one root if $a < b$ and an infinite set of roots where $x \varepsilon R$ and $x \geq a$ when $a = b$.
- Equations of the form $|x + a| = |x + b|$ will have only one solution.
- On the other hand, $|ax + b| = cx + d$ has one solution if $a < c$. However, no conclusive generalisation was obtained on the effect of b and d in the case where $a > c$. The few examples looked at seemed to vary from question to question.

In general, participants began to appreciate the difference between the equations $|x + 2| = 2x + 1$ and $|2x + 1| = x + 2$.

## 6. Conclusions

There is a wide research base that concurs with the findings of this study that technology should be embedded within the teaching and learning of mathematics. The use of GeoGebra allowed participants to develop a deeper understanding of equations involving modulus functions. It aided understanding and enabled problems to be addressed that would have been impractical or inefficient to tackle.

The selected examples highlighted the varying errors that learners make when solving equations involving modulus functions. The use of multiple representations helped participants to investigate why the algebraic methods sometimes yielded spurious answers.

Participants were exposed to multiple representations of modulus functions. For example, $f(x) = |x - 2|$ is the same as $f(x) = \begin{cases} x - 2 \text{ for } x > 2 \\ -x + 2 \text{ for } x < 2 \end{cases}$. This was complimented by the visual representation of the same function on the GeoGebra screen, hence consolidating the link between the two algebraic representations.

There is a drive by government to embed technology in the teaching of mathematics. This study focused on equations involving modulus function with a small number of participants. The impact of technology on the teaching and learning of mathematics cannot be generalised from this study. Further research focusing on the impact of technology needs to be done. There is need to embed multiple representations in the teaching of this topic, to enhance the understanding of what modulus functions are.

Participants should be encouraged to sketch graphs first, to see the number of solutions and identify where they are located. However, Yerushalmy's (1991) research suggests that appreciating the links across multiple representations is not automatic. Yerushalmy (1991) found that even after extensive experience with multi-representational learning, designed to teach understanding of functions, only twelve per cent of participants gave answers that involved both numerical and visual representations. Yerushalmy observed that most answers reflected the use of one representation and a neglect of the other.

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